The Arithmetic Teacher

MARCH • 1958

To Tell—or Not to Tell
CAROLINE HATTON CLARK

Commercial Games for Arithmetic
DONOVAN A. JOHNSON

Twenty Methods for Improving Problem Solving
ROBERT H. KOENKER

Divisibility and Prime Numbers
PAUL YEAROUT

Casting Out Nines and Other Numbers
LUCY E. DRISCOLL

THE ARITHMETIC TEACHER

a journal of
The National Council of Teachers of Mathematics

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All editorial correspondence, including books for review, should be addressed to the Editor. Advertising correspondence, subscriptions to THE ARITHMETIC TEACHER, and notice of change of address should be sent to:

The National Council of Teachers of Mathematics
1201 Sixteenth St., N.W., Washington 6, D. C.

Entered as second-class matter at the post office at Washington, D. C., June 21, 1954, under act of March 3, 1879. Additional entry granted at Menasha, Wis.

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THE ARITHMETIC TEACHER is published six times per year, in October, November, December, February, March and April. The individual subscription price of $3.00 ($1.50 to students) includes membership in the Council. Institutional Subscription: $5.00 per year. Single copies: 50¢ each. Remittance should be made payable to the National Council of Teachers of Mathematics, 1201 Sixteenth St., N. W., Washington 6, D. C. Add 25¢ per year for mailing to Canada, 50¢ per year for mailing to other foreign countries.

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THE ARITHMETIC TEACHER

Volume V Number 2

March 1958

To Tell—or Not to Tell

CAROLINE HATTON CLARK

New Hope, Pennsylvania

IF THE PUPIL is to get optimum benefit from his arithmetic instruction, it is essential that the teacher distinguish between what elements of a topic the child learns by being **told** and what elements he learns by being guided to analyze, to discover, to **deduce**. This paper presents 15 items that enter into the over-all development of the number *six*. (Similar items would apply in the development of any other one-digit number.) The reader is invited to classify each of these items. When the reader encircles the "D" placed after an item, it will mean that he considers this to be a phase of learning he would help the child arrive at mainly by DEDUCTION. On the other hand, if he considers a learning item to be one that is dictated by cultural usage, is arbitrary, and one that the learner cannot deduce, the reader can so indicate by encircling the "T" for TOLD after the item. Some items may rate both a "D" and a "T".

A table is given at the end of this article indicating the writer's classification of each item. The table is not to be regarded as a score card, but the reader may wish to see how much agreement there is between his own classifications and the writer's.

Learnings Associated with the Number Six

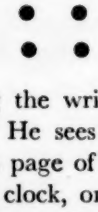
1. Almost all children enter school with initial learning about six; they have heard and spoken the word "six." They probably have been taught to do rote counting to ten or beyond—that is, they can say number words in order without regard to groups of objects. When they say the word "six," or any other number-word in such counting, it has no meaning. They may also have used the word "six" in rhymes, songs, or games. And of course, for all children, "six years" is the magic age when the school door opens, but they have little comprehension of the meaning of being "six years old." To most children it is the birthday that comes after one has been five years old for quite a long time. **T** or **D**?

2. Rational counting to ten and beyond comes next. In this counting, the child points to objects in order as he says the number names, realizing that the last number named tells the number in the group. At this point, 6 apples, or 6 pennies, or 6 children, or 6 counters, 6 dots, or 6 of anything, take on meaning. The child is guided to abstract the common quality of such groups —that is the *number* in the group. **T** or **D**?

3. In learning to count, he also begins to recognize the series aspect of six. He senses that a group of six always has one more in it than a group of five, and one less than a group of seven. This learning builds readiness for understanding the processes of addition and subtraction. **T** or **D**?

4. Simultaneously the child is becoming able to recognize groups of objects up to six without counting. He first recognizes one, two, or three objects in any arrangement of pattern. Four, five and six objects are most easily recognized without counting in these patterns:

or



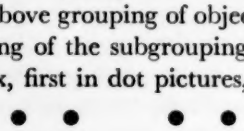
T or **D**?

5. Recognizing the written symbol "6" may come next. He sees the figure 6 all about him—on a page of his book, on the calendar, on the clock, on a price tag, on the television dial, etc. Gradually, he comes to associate it with a corresponding group of objects, and to assign it to its proper place in a series with 5 and 7 as its nearest neighbors. **T** or **D**?

6. Following closely may come the writing of the symbol "6". **T** or **D**?

7. A "taking apart" and "putting back together" of a group of six objects is a usual next step. The pupil discovers, and verbalizes the discovery, that 6 objects may be separated into groups of 1 and 5, 2 and 4, 3 and 3, or 2 and 2 and 2. Almost simultaneously he comes to appreciate that when he puts these subgroups back together again he always has a group of 6. **T** or **D**?

8. The above grouping of objects leads to the recording of the subgroupings within a group of six, first in dot pictures, as



and then in symbols: 3 and 3 are 6. **T** or **D**?

9. About this time, the pupil may learn to read the word symbol "six." **T** or **D**?

10. The skill above is closely followed by learning to spell and write the word "six." **T** or **D**?

11. As the child progresses, many common social uses of six begin to impress him: a nickel and a cent make six cents; a half dozen eggs is six eggs; soft drinks come in cartons holding six bottles; etc. **T** or **D**?

12. As a result of many activities such as those described in points 7 and 8 above, the pupil *discovers* important number relationships like the following:

- 5 and 1 are 6, so 1 and 5 are 6.
- If I take 5 from 6, I have 1 left; if I take 1 from 6, I have 5 left.
- 3 and 3 are 6, so 3 and 2 are 5.
- If I take 3 from 6, I have 3 left; if I take 4 from 6, I have only 2 left. **T** or **D**?

13. Now the child is ready to be taught some arithmetic vocabulary (words standing for arithmetic ideas). He learns to associate the word "*add*" with "counting on" or "putting together" of groups or numbers, and finding how many there are all together.

He discovers that when he needs to find the answer to such questions as these, he can do so by *adding:*

- John has 4¢. Ted has 2¢. How many cents have they together?
- Bill had a cent. He found a nickel. How much money had he then?

The word "subtract" will be associated with finding the answers to such problem situations as:

- Finding how many are left?
  Jim had 6¢. He spent 4¢. How many cents did he have left?
- Finding how many more are needed?
  Tom wants to buy a 6-cent whistle. He has 4¢. How much more does he need?
- Finding how many are gone?
  Ned had 6¢. He lost some. He has only 4¢ left. How much did he lose?
- Finding the other number.
  There are 4 children in the Clark family. Four are boys. How many girls are there?

In each of the above situations, the pupil should recognize that he is dealing with:

(a) *A known whole group*
(b) *A known subgroup*
(c) The other subgroup; the *one he wishes to find*

The unknown subgroup is found by *subtracting* the known subgroup from the known whole group. It is productive for the learner to be given practice in identifying these three elements in varied problem situations. Learning aids such as the following help the child sense the need and convenience of subtraction:

As children count, drop one-by-one, 6 counters into a box. Reach into the box, and withdraw 2 counters. Show the children the 2 counters, and ask, "How can we find the number left in the box without counting them?"

A variation of this practice is to withdraw in a closed hand an unspecified, unknown number of counters. Then have a child look into the box and count the number left. Ask, "How can we find how many counters I have in my hand without counting them?

The learner will also come to realize that he can subtract to find the difference, as in:

Joe has 6¢. Sue has 4¢. How much more has Joe than Sue? How much less has Sue than Joe?

In situations like this, he mentally superimposes Sue's 4¢ on his 6¢; sees that 4¢ as a "known part" of his 6¢, and subtracts to find the "unknown part," (the difference) thus obtaining answers to the question, "How much (or many) more?" or "How much (or many) less?"

0 0 0 0 0 0

0 0 0 0

At a later date, his vocabulary will be similarly extended to include the words "multiply" and "divide" in association with 3 twos and 2 threes, 6 ones and 1 six. **T** or **D**?

14. To facilitate written records of thinking, arithmetic symbolism and algorisms (forms of writing examples) are introduced next. The child learns that this sign (+) tells him to add. 4+2 are 6 means 4 and 2 are 6. The sign (−) tells him to subtract. He is taught to read 6−4 is 2, as 6 take away 4 is 2.

Likewise 4 says to him, "Add the 4 and
$\underline{+2}$ the 2"

And 6 says to him, "Subtract the 4
$\underline{-4}$ from the 6."

At a later date the signs +, −, = take on the names of plus, minus and equals in expressions such as 5+1=6, and 6−5=1. **T** or **D**?

15. The child learns to use the symbolism and algorisms learned above in the solution of problems. For example he writes 2 to record his thinking in finding the answer to this problem:

$$\begin{array}{r} 2 \\ \underline{+4} \\ 6 \end{array}$$

Jane has 2 big dolls and 4 little dolls. How many dolls has she in all? **T** or **D**?

Practicing and Testing for Recall

By this time in his study of "six" the child has discovered and rediscovered, applied and related the following addition and subtraction facts with sums and minuends of 6:

$\begin{array}{r} 5 \\ \underline{+1} \\ 6 \end{array}$	$\begin{array}{r} 1 \\ \underline{+5} \\ 6 \end{array}$	$\begin{array}{r} 4 \\ \underline{+2} \\ 6 \end{array}$	$\begin{array}{r} 2 \\ \underline{+4} \\ 6 \end{array}$	$\begin{array}{r} 3 \\ \underline{+3} \\ 6 \end{array}$
$\begin{array}{r} 6 \\ \underline{-5} \\ 1 \end{array}$	$\begin{array}{r} 6 \\ \underline{-1} \\ 5 \end{array}$	$\begin{array}{r} 6 \\ \underline{-4} \\ 2 \end{array}$	$\begin{array}{r} 6 \\ \underline{-2} \\ 4 \end{array}$	$\begin{array}{r} 6 \\ \underline{-3} \\ 3 \end{array}$

It is now time for him to determine whether he has memorized these facts of 6, so that they will function most effectively in problem solving and in more complex computations. When he sees or hears "4 and 2," his immediate response should be 6, without recourse to counting or other discovery methods. For the first time he now drills for recall of facts. He should be tested frequently and helped to keep a record of those facts on which he needs more practice. He should be stimulated to reach his goal of mastery of the facts by competition with his past performance rather than by competition with the work of other pupils.

WRITER'S CLASSIFICATION OF ITEMS 1–15

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
T	T & D	D	T & D	T & D	T	D	D	T	T	T & D	D	T & D	T	D

Postscript

Unfortunately, many teachers use the "T" approach for aspects of learning, when the "D" approach could be used. They believe that the "T" approach is more economical of time for both the learner and the teacher. It is the writer's opinion, however, that the alleged economy is invalid. In situations where the "D" approach can be and is employed the resulting learning is more enjoyable, more permanent and leads to greater resourcefulness in mathematical thinking.

EDITOR'S NOTE. The traditional method of instruction is centered largely in "telling" instead of in "thinking" and "deducing." Actually it matters little whether the "telling" is done by a teacher, a textbook, or some other pupil. "Telling" has its place, as Mrs. Clark points out, but it is apt to sacrifice one of the primary aims of education and that is to teach people to think for themselves. It is just as important to practice thinking as it is to practice for memory of a pattern in computation. And the place to begin thinking is in the very lower grades. A high order of insight into a teaching situation is demanded in order to know when to "tell" and when not to "tell" and this ideally applies to individuals and not to groups of children. Frequently, the best help a teacher can give is the least help that is necessary. We may not agree totally with Mrs. Clark's judgment as to which items are basically "telling" items but we must respect her point of view when she urges that we do much more "deducing" with out pupils. There are levels of deduction, at certain stages it is desirable to use the Socratic method while at other times the steps should not be made obvious to the pupil. Let us remember to encourage children to practice thinking.

Reading Numbers Modernized

"Four and twenty blackbirds baked in a pie" reveals a form of number which is a bit archaic and no longer in use except from thirteen to nineteen. However, in other languages this old form is still good usage and seems not in question. In German one meets still, "vier und zwanzig", "drei und achtzig", as well as "dreizehn" and "vierzehn" for 24, 83, 13 and 14, respectively.

In modern English we have changed all numbers from 20 to 100 to the normal order of reading, the tens digit first and all digits in order. However, we are still archaic in the teens, since here we reverse the order. For example, in 16,814 we read sixteen thousand 8 hundred fourteen. In ancient times 26,824 would have been similarly read as six and twenty thousand 8 hundred four and twenty. How ridiculously awkward! Let's modernize our "teens" too!

To modernize our "teens," instead of reading 16 as sixteen, it should be read teen-six, just as four and twenty is now read twenty-four, omitting the "and." We would then count: nine, ten, teen-one, teen-two, teen-three, etc.—to teen-eight, teen-nine, twenty, twenty-one, etc. Then 16,814 like 26,824 would be read teen-six thousand eight hundred teen-four. Not only would such a modernization of our reading numbers make sense logically, but teaching numbers would be simplified too.

Contributed by

H. C. CHRISTOFFERSON

Commercial Games for the Arithmetic Class

DONOVAN A. JOHNSON
University of Minnesota, Minneapolis

"AMUSEMENT AND PLEASURE ought to be combined with instruction in order to make the subject more interesting. There should be games of various kinds such as a game played with different kinds of coins mixed together. There should also be problems connected with boxing and wrestling matches. These things make a pupil useful to himself and more wide awake."—Plato

Even as Plato proclaimed, *some* learning of arithmetic can be accomplished through games. Games are usually considered a recreational activity. They are a means for releasing boredom or tension. Games break down barriers between strangers and quickly establish a friendly group spirit. At the same time games may be a way to learn to follow rules, to be cooperative, to be observant, or to practice diligently. Games also give participants a chance to develop restraint, to contribute as a leader and follower, to accept responsibility for individual as well as group action. However, learning arithmetic is not all a game. Other types of individual and independent group activities are obviously essential. But the variety and activity of a game may break up the monotony so frequently present in practice or drill lessons.

Unquestionably the key to learning arithmetic is through meaningful experiences and practical applications. However, the skills of computation need to be nurtured by a variety of systematic practice and drill. Thus the total approach to learning arithmetic uses meaning, practice, and application. The meaning of numbers, the understanding of a process, the mathematical structure involved precedes the practice. Practice then is the part of the learning process which builds accuracy, efficiency, and retention. Purposeful practice in the right amount at the right time will help build the arithmetical competence that business, science, industry, and education are demanding. Arithmetic games are ideally suited for a systematic program of practice. However, games can serve a variety of functions in your classroom, such as:

- Build desirable attitudes towards arithmetic
- Provide for individual differences
- Provide appropriate "homework" for parent-child activity
- Make practice periods pleasant and successful

Right now there are a great variety of arithmetic games available from commercial publishers. Most of these games are available at school supply stores or toy stores. They are usually attractive and durable with complete playing instructions included. Of course you can make up a variety of games with flash cards or number cards. However, the convenience of commercial cards makes them highly desirable. Try them sometime. You will be pleasantly surprised with the vigor pupils will exhibit in learning combinations in order to win a game instead of working problems to win the teachers favor.

Commercial Arithmetic Games

PRIMARY NUMBER CARDS AND GAMES..... $1.00

This set consists of sheets of cardboard with numbers and number patterns printed on them. The cardboard is to be cut into cards and made into games for self testing of addition and number concepts.

HALL AND MCCREARY CO.

COMBINATIONS ARE FUN..... $1.00

This game consists of 8 brick houses, each house has 13 windows. Addition problems are on the windows and the answers in the house.

PALFREYS SCHOOL SUPPLY CO.

K RECT..... $1.10 each game

1. Set A. Addition and Subtraction
2. Set B. Division and Multiplication

The game has 60 cards, 15 books of four cards each. A book consists of a fraction, a decimal, a per cent, and a division all having the same value. The object of the game is to complete as many books as possible.

SELF-TEACHING FLASHERS

SELF-TEACHING FLASHERS..... $1.10 each set

1. Set A. Addition and Subtraction
2. Set B. Division and Multiplication

The 81 cards for this game have combinations of numbers for games with groups of 3 to 10 pupils. The pupil giving the right answer of the flashed card, receives that card. The number on the top of the card is always the answer to the combination on the reverse side of the card.

SELF-TEACHING FLASHERS

FUN WITH NUMBERS, I WIN..... $ .75 per deck

1. Addition for ages 5–7
2. Addition for ages 6–8
3. Subtraction for ages 5–7
4. Subtraction for ages 6–8
5. Addition for ages 7–9
6. Addition for ages 8–10
7. Subtraction for second grade
8. Multiplication for ages 9 and up
9. Short Division for ages 9 and up
10. Multiplication for ages 10 and up
11. Short Division for ages 10 and up
12. Short Division for ages 10 and up
13. Multiplication for ages 10 and up
14. Multiplication for ages 10 and up
15. Multiplication for ages 10 and up
16. Short Division for ages 10 and up
17. Short Division for ages 10 and up
18. Short Division for ages 10 and up

Each deck consist of 25 pairs of cards. Each pair consists of a card with a problem in black and a card with a numeral answer in red. The object of the game is to match correctly the number combinations and answers.

EXCLUSIVE PLAYING CARD COMPANY

ARITHMETIC DOMINOES..... $1.25

Each domino of this game has a number on the blue side and a matching addition fact on the reverse red side. The number on the blue side is given as a figure, a series of dots, and as a word. The addition fact on the red side is given in dots with the answer as a figure and a word. Directions are given for five games which give practice in counting, adding, writing numbers, reading numbers, making number families and representing addition facts visually.

THE ARITHMETIC CLINIC

NUMBERLAND SPECIAL (Addition facts)..... $1.50 each game

OLD WOMAN IN THE SHOE (Subtraction facts)

PARKING LOT (Multiplication facts)
ROUND-UP (Division facts)

These games consist in matching number combinations with the answers. The answers are dressed up to represent train cars, children, autos or cattle so as to stimulate child interest.

IDEAL SCHOOL SUPPLY CO.

MY ARITHMETIC TEACHER..... $1.15

The number combinations for addition, subtraction, multiplication and division are given on a cardboard base. The answer is made concrete by having the player put pegs in holes associated with the answer to a number combination.

THE ARITHMETIC CLINIC

SAY-IT..... $1.25 each set or $4.50 for complete set

(1) *Addition Game*
(2) *Subtraction Game*
(3) *Multiplication Game*
(4) *Division Game*

Each game contains large cards each containing 16 number facts. Smaller cards contain all the number facts on the larger card. The game is played like Lotto with the smaller cards being claimed by stating the number fact and covering this fact on the players large card.

THE GARRARD PRESS

QUIZMO..... $2.00 each set

(1) *Addition and Subtraction*
(2) *Multiplication and Division*
(3) *Fractions*

This game is a variation of Bingo to provide practice in number combinations when a Quiz card with a number combination is selected at random and held up, the players cover the current answers on their cards with a marker. Each set contains 41 master cards, 126 Quiz cards, and markers.

MILTON-BRADLEY CO.

IMMA WHIZ..... $1.00 each game

(1) *Addition and Subtraction*
(2) *Multiplication and Division*

This is another variation of Bingo in which the call cards are numbers representing answers to number combinations. The players cover the problem for which the caller gives the answer

KENWORTHY EDUCATIONAL SERVICE

SCOR-O..... $1.50 each set

(1) *For Addition*
(2) *For Subtraction*
(3) *For Division*
(4) *For Multiplication*

Each set contains four boards, one for each player, and 108 number blocks. The number blocks are used to complete problems on the pupil boards. The score for the game is the sum of the correct answers.

CHAMPION PUBLISHING CO.

ARITHO..... $1.75

This is an arithmetic Bingo game in which the players cards have numbers that are answers to problems on call cards.

PSYCHOLOGICAL SERVICE

SPINNO

Dials with number combinations for addition, subtraction, multiplication or division may be mounted on a cardboard base. As the player gives the answer to a combination he can turn a dial with a slot to check whether he has given the correct answer.

JOHN C. WINSTON COMPANY

IMOUT ..... $5.25 each game

(1) *Addition and Subtraction*
(2) *Fraction*
(3) *Multiplication and Division*

These games are similar to Bingo. Play consists in matching answers on the playing card with fractions or multiplication or division problems drawn at random. A spinning device is included to lend variety to the method of selecting problems.

IMOUT

SPINNER FRACTION PIE GAME .... $2.95 each game

SPINNER FRACTION SQUARES GAME

Six 3-inch rubber fraction pies or squares cut in halves, thirds, fourths, sixths, and eights serve as parts to make wholes. Fractions are written on sections of the spinner which is used to select fractional parts.

CREATIVE PLAYTHINGS

ADDO ..... $1.50 each game

MULTO

These games resemble bingo in that the players have cards with nine numbers which represent sums or products. The problems are on call cards which a Banker selects at random.

KENWORTHY EDUCATIONAL SERVICE INC.

COMBINATIONS ARE FUN $1.00

There are eight "brickhouses" in a set, each house containing 13 windows. An addition problem is printed on the outside of each window. The answer to the problems is given inside the house when the window is opened.

KENWORTHY EDUCATIONAL SERVICE INC.

HIT $1.50

This set contains charts and markers for 25 players. Games for group play are based on the multiplication and division facts.

THE PLAWAY GAMES

SUM FUN $1.50

Numbers of ten multi-colored cubes are used to build runs of consecutive numbers and sets of like numbers. The scoring requires the addition, subtraction or multiplication by one digit numbers.

KRAEG GAMES

TIMES SQUARE ..... $2.75 set of four games

This game consist of call cards, markers, and game cards similar to those used in Bingo games. There are four types of call cards which make it possible to play four games namely Times Square, Divvy-Up, Add-A-Lad, and Anchors Away. As indicated by these titles the games involve the combinations for multiplication, division, addition and subtraction.

KRAEG GAMES

MAKE ONE $1.00

To "Make One," the players put together fraction cards to percent cards, or both combined, to make a whole unit. Each card has a section of a circle shaded to represent the value of the fraction of per cent written on the card.

THE GARRARD PRESS

ADDI-FAX ..... $1.50 each game

MULTI-FAX
FRACTI-FAX

The object of these card games is to make "faxs" or "books" containing three cards. In Addi-Fax these books are two addends and the sum; in Multi-Fax, two factors and the product; in Fracti-Fax, a fraction, a decimal and a per cent all three of which are equivalent.

THE PLAWAY GAMES

THIRTEEN $2.00

Number Tiles and a Playing Board similar to Scrabble are the pieces used for this game which acquaints children with essential arithmetic tools while they play. Play consists of the most advantageous placing of Tiles in combination with others on the Board to total a multiple of some number selected by the players. Thus, the game gives practice in addition and multiplication.

CADACO-ELLIS, INC.

MATH MAGIC $3.00

MATH MAGIC features 3 games EQUALIZE, ANALYZE, and MINIMIZE which involve mathematical thought processes, Number Cards, Symbol Cards, and numbers pre-determined by an Indicator Dial showing all of the 53 Factorable Numbers, which can be obtained by multiplying Factors 2 through 12, are used interchangeably. EQUALIZE, ANALYZE AND MINIMIZE promote skill in addition, subtraction, multiplication and division for the imaginative manipulator of numbers.

CADACO-ELLIS, INC.

ADD-A-GRAMS $1.00

This game consists of number blocks and plus blocks. The game consists in arranging the blocks to build sums in cross-words fashion.

PALFREYS SCHOOL SUPPLY CO.

FRACTION FUN $1.50

The pieces for this game are circles or fractional parts of circles. The game consists in calling for fractional parts of different denominators to complete one whole unit circle.

PALFREYS SCHOOL SUPPLY CO.

NUMBER FUN $1.25

This game consists of 150 interlocking cardboard tiles and four racks to hold the tiles. The players build a series of arithmetic problems by adding interlocking numbered tiles.

PALFREYS SCHOOL SUPPLY CO.

LOSE YOUR SHIRT..... $2.95

Somewhat similar to Scrabble, this game uses card board tiles with numbers and computation symbols. The object of the game is to build computation examples that will combine to give the player a very large scrore. Each player has his own game sheet on which to build his examples.

THE QUANTUM CORP.

ELECTRIC QUIZBOOK

This is a simple electrical device that flashes a light when a question and its answer are touched simultaneously with pointers. The cards with the problems and matching answers may be written to fit any topic being studied. The wiring is readily charged without tools so that the order of the answers may be varied. It is operated with a single flashlight cell and is completely safe. Wiring the board is a simple exercise in electric circuits. The complete set includes all parts and an instruction handbook.

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St. Louis, Missouri

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Pennsylvania

EXCLUSIVE PLAYING CARD COMPANY
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GARRARD PRESS
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Champaign, Illinois

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IDEAL SCHOOL SUPPLY CO.
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Using Arithmetic Games in the Classroom

The success of an arithmetic game, like any classroom material or technique, is highly dependent on how it is used. If an arithmetic game is to serve a real function it should be used at the right time, for the right purpose, in the right way.

1. *Select a game according to the needs of the class.* The basic criteria is that the game make a unique contribution to learning of arithmetic that cannot be attained as well or better by any other material or technique. The material of the game should be closely related to that of the regular classwork. Whatever game is selected should involve important skills and concepts. Major emphasis should be on the learning of concepts or skills rather than on the pleasure of playing the game itself.

2. *Use the game at the proper time.* If games are to make the maximum contribution to the learning of arithmetic they should be used at the time when the ideas or skills are being taught or reviewed. However, many teachers prefer to use games after completing a topic, on the day before a vacation, or during the pupils free time at noon, in study periods or homerooms. Others use them during days of heavy absence due to storms, concerts, or excursions. Some teachers use them as rewards for work well done while others use them for remedial work. Usually games should be played a relatively short time so that pupils do not lose interest.
3. *Arrange the game situation so that ALL pupils will be participating in every play.* Even though only one person is working on a certain problem at a given time every team member must be responsible for its solution too. Games must also avoid embarrassing the person who cannot solve a problem. Keep comments positive, commend good work rather than making unfavorable comparisons. Whenever possible pupils should compete with their peers, and should be working on material according to their ability.
4. *Plan and organize the game carefully so that the informality and excitement of the setting does not defeat its purpose.* Teach the playing of games in a planned, organized way as you would present other activities. Have all the materials at hand so that the game can proceed in an orderly fashion. When a new, complex game is being played, start with a few essential rules and then add other rules as they are needed during the game. Use a few practice plays to help get started. Then expect good work, as neat and accurate as regular classwork. Pupils may referee the competition as well as play the game. Before beginning the game the participants should be instructed on the purpose of the game, the rules of the game, and the way to participate. Often the pupils can establish ground rules, so that everyone (including the teacher) may enjoy the activities. "Coaching" or "kibitizing" should never be allowed. The loss of points for the breaking of rules will usually be sufficient to maintain appropriate behavior. Avoid the choosing of team members by pupil captains so that low ability pupils will not be embarrassed by being last choice.
5. *Emphasize the responsibility of learning something from the game.* Follow-up activities such as discussions, readings, or tests will emphasize this responsibility. As the teacher, evaluate the results by asking yourself how successful the game was in promoting desired learning.

EDITOR'S NOTE. Donovan Johnson has pointed out the values that can be found in good arithmetic games. He also says that games alone will not suffice to teach arithmetic. He has listed many of the commercial games that may be obtained. He does not evaluate them. Much depends upon how a game is used. In the hands of one group of pupils it can be a waste of time while with others and with the guiding spirit of a good teacher much of value can be derived. For most normal youngsters, the study of arithmetic should be interesting, enjoyable, and respected. A good game can be a powerful stimulant. But games need not be commercial: many good games are adaptations developed by teachers. Put a little fun in your teaching, it may pay big dividends.

Grades 1 to 3

The Montclair, New Jersey public schools have printed an interesting booklet which teachers give to parents during the second report period which takes the form of a conference between parent and teacher instead of a regular report card. In twelve pages, the primary grade child is characterized and the main goals in each of the subject areas are listed. The schools want the parents to know what is expected of pupils in these grades and have found this to be a good method of giving the information. Readers may obtain a copy for 15¢ by writing to Robert E. Gitelman, 22 Valley Road, Montclair, N. J.

Twenty Methods for Improving Problem Solving

ROBERT H. KOENKER
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THE FOLLOWING SUGGESTIONS for improving problem solving in Arithmetic have proven to be of value by research and/or classroom practice.

1. *Use the Whole Method of Attacking Problems.* There are several different methods of attacking a problem; however, in general it is best for the child to read through the problem and understand it as a whole first rather than as a series of parts. When using the traditional formal analysis method of solving a problem, a child reads the problem so analytically that he may miss the whole point of the problem. Formal analysis stresses the facts and figures before the child understands what the problem is about. The outmoded cue word approach to problem solving whereby the child looks for the word that tells him to add, subtract, multiply, or divide is a mechanical nonmeaningful approach to solving a problem. The recommended whole approach leads to understanding and thinking better than the formal analysis or cue word approach. However, with some children these latter two approaches to solving a problem may prove of some value in some instances.
2. *Estimating Answers to Problems Before Solving.* Estimating the answer to a problem before solving it causes the child to think through the problem. Estimating the answer also reduces the chances of the child's obtaining a "wild" answer, and gives him the necessary experience he needs in working with round numbers. Furthermore approximately 85 per cent of the arithmetic problems adults face in their everyday experiences are solved by estimating and not by paper and pencil. Frequent exercises in estimating answers to computations, such as follow, will definitely aid the child in estimating answers to arithmetic problems.

   ESTIMATE THE ANSWER:

   (1) 19×5 is about: 90, 100, or 110? _____

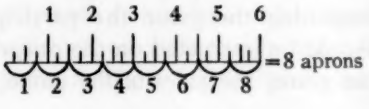
   (2) \$5.98−\$2.19 is about \$3.00, \$4.00, or \$5.00?_____

   (3) 72−37 is about: 50, 40, or 30? _____

   (4) 38¢×6 is about: \$2.00, \$1.80, or \$2.40?_____

3. *Diagramming Problems:* Diagraming a problem many times will clarify the problem in the child's mind, as well as to help him solve it. Furthermore, if a child can properly diagram a problem, this will show the teacher that he understands what he is doing. It must be realized that diagraming a problem is not necessarily the most mature method of solving a problem, but with many children it is a necessary step toward such maturity. Following are two examples of solving problems by diagraming:

   (1) If an apron takes $\frac{3}{4}$ yard material, how many aprons can be made out of six yards of material?



(2) Carl had 15 marbles. He gave 7 marbles to his brother Bill; how many marbles did Carl have left?

4. *Dramatizing Problems.* Many times it is of value to dramatize problems. Dramatizations can be made by the teacher, pupils, or both. Such dramatizations are not only of great interest to children, but also make the problems more realistic. Problems, dealing with units of measure; money; and objects such as marbles, spools, and beads lend themselves readily to dramatizations.
5. *Orally Solving Problems.* The oral solution problems as a class or group exercise each day for ten minutes is a very worthwhile technique for improving problem solving. In solving oral problems the child must size up the problem as it is read to him. He cannot keep referring to it, and since paper and pencil are not used he must focus his thinking on the problem and come up with a solution. Oral problems must include numbers that are small enough for mental computations. However, some oral problems could be worked by rounding off numbers so as to get an estimated answer only. Following are illustrative problems that could be worked readily without the aid of paper and pencil:
   (1) Carl had $1.50; how many 3¢ stamps could he buy?
   (2) Bill is 8 years old. He is four times as old as his brother. How old is his brother?
   (3) Twenty-five boys and thirty girls were at the picnic. How many children were there in all?
6. *Encourage Children to Work Problems Using Different Methods.* Working problems using different methods causes a child to think quantitatively. If the child works problems using different methods it will also tend to show him the interrelationships that exist in our number system. For example, multiplication problems can be solved by addition, and division problems can be solved by subtraction. A child will eventually want to solve a problem by the shortest correct procedure, but again a child should understand that some problems may be worked in more than one way. Furthermore, one method of solution may have meaning to one child, while another method of solution may be more meaningful to another child. To illustrate, the following problems may be worked by three different methods, all of which are correct:

   PROBLEM: If apples were selling at 3 for 10¢, how much would 9 apples cost?

   Method 1: Find the cost of each apple and multiply by 9.

   $10¢ \div 3 = 3\frac{1}{3}¢; \quad 3\frac{1}{3}¢ \times 9 = 30¢$

   Method 2: If 3 apples cost 10¢ then 9 apples would cost three times as much or 30¢

   Method 3: Finding the answer by proportion.

   $3{:}10 \text{ as } 9{:}x, \quad 3x = 90, \quad x = 30$

7. *Differentiating Problems for the Various Ability Levels.* Numerous studies have shown that there are wide differences in arithmetic ability at any grade level. Therefore, it is advisable to differentiate or individualize instruction in problem solving as much as possible. No doubt there will be some problems for all to work, but in general, problems should be differentiated as to difficulty so as to meet the various ability levels within a class.
8. *Making up Problems.* It is many times necessary to supplement the problems found in a book by having the teacher and/or pupils make up arithmetic problems. Such problems should be realistic and could grow out of class activities in science, social science, hygiene, etc.

Many times the children will want to make up original problems dealing with their out of school experiences. Problems of this type could deal with buying at the store, playing games, taking a trip, etc. A good arithmetic program should also include units of instruction such as "Telling Time," "Our Weather," "Healthful Foods," "The Air Age," etc. Many problems will grow out of such units of instruction.

9. *An Understanding of Arithmetic is Prerequisite to Problem Solving.* It is obvious that many children fail at problem solving in arithmetic because they lack the arithmetical background necessary to do the required computations. A child who is skillful in arithmetic and understands the number system and its interrelationships will be more successful in problem solving than a child who lacks these understandings. Furthermore, a child who is skillful in arithmetic can devote his time to the problem itself since he understands how to solve the needed computations.
10. *Using Concrete Objects and Devices in Solving Problems.* Such aids as play money, tongue depressors, bead counters, abaci, fraction boards, etc., are invaluable to the child in finding the answers to problems. These aids and devices not only add meaning and understanding to arithmetic, but also greatly stimulate and motivate the child.
11. *Going Over Assigned Problems with Children.* It seems advisable that the teacher should read and discuss with the children certain of the assigned arithmetic problems before the pupils attempt to work them. In one study it was found that such a procedure proved of significant value, particularly for the average and below average students.
12. *Checking or Proving Answers.* Besides estimating the answers to arithmetic problems it is many times advisable to have the child check his work. It is also of value to have the child prove his work either by using another method, or by using concrete aids and devices. Checking answers reduces errors. Proving answers shows that the child understands what he is doing.
13. *Working Problems without Numbers.* In working problems without numbers the child tells what method(s) would be used to solve the problem. Here the child is not distracted by numbers and can focus his attention on the method of solution needed. A child can do many of these problems in a short period of time. Following are illustrative problems:
    (1) Jim knows how much money he earned in four weeks. How would he find his average weekly earnings?
    (2) Jane knows how many yards of ribbon she needs and the cost of each yard. How can she find how much the ribbon will cost?
    (3) Bob knows how many marbles are in each of two boxes. How would he find how many marbles he has in all?
14. *Solving Problems with Irrelevant Facts.* Having children solve problems with irrelevant facts causes them to examine critically all the details before solving. Sometimes the children may be asked to find the irrelevant facts, other times they should be asked to solve the problem and in so doing avoid the irrelevant facts. Following are illustrative problems that contain irrelevant facts: Underline the irrelevant fact in each problem:
    (1) A farmer had 120 chickens. He sold 80 of them at $1.40 each. How much money did he receive?
    (2) Bill had 36 marbles, Jim had 24 marbles, and Carl had 30 marbles. How many marbles did Bill and Carl have together?
    (3) Mary had $10.00. She bought a a pair of shoes selling originally for $6.00 on sale at "1/3 off." How much did she pay for the shoes?

15. *Finding Missing Facts in Problems.* Give children problems to solve that require other facts or data before they can be solved. Such problem exercises cause the child to do a higher type of thinking than is required by the typical type of arithmetic problem. The children may be asked to give the facts needed that are not included in the problem, or they may work problems of this type by supplying the missing facts first. Following are illustrative problems that have missing facts:
Find the missing facts:
(1) Bill bought 3 yards of canvas at 35¢ a foot. How much did the canvas cost?
(2) Mary is 4 feet tall. Her sister, Annie, is 44 inches tall. How much taller is Mary?
(3) Jeff sold 4 bushels of tomatoes at 75¢ a peck. How much money did he take in?

16. *Finishing Incomplete Problems.* Another way to improve problem solving is to give children incomplete problems which they are to finish. Following are illustrative examples of incomplete problems which the children are asked to complete and then solve:
(1) Frank had $20.00. He bought a pup tent for $14.95.
(2) There are 12 months in a year. Susan saved 50¢ a month.
(3) There are 36 children in the third grade, 29 children in the fourth grade, and 41 children in the fifth grade.

17. *Stressing Careful Reading of Problems.* If a child does not read well or is a careless reader, he no doubt will experience difficulty in solving arithmetic problems. It is necessary for the teacher to stress reading during the arithmetic period. However, the reading skills required to read arithmetic problems are different from the reading skills required for ordinary recreational reading. Stress should be placed on the careful, fact finding type of reading needed in reading problems during the arithmetic period.

18. *Developing an Understanding of Arithmetic Vocabulary.* Studies show that a good understanding of arithmetic vocabulary and terminology is definitely related to success in problem solving. It is best that a child learns the vocabulary and terminology of arithmetic while working problems; however, some vocabulary exercises will prove of value. Following are some illustrative exercises in arithmetic vocabulary or terminology:
(1) To find the sum of two numbers means to: subtract, add, multiply, or divide?
(2) To find out how many times one number is contained in another number we: add subtract, multiply, or divide?
(3) Which number is the product? $12 \times 8 = 96$
(4) What does C.O.D. mean?
(5) Write the sign for per cent.

19. *Stressing Neatness of Work.* An analysis of children's computational errors would show that many errors are made because the computations are not neatly arranged. The following division error was made due to careless arrangement of numbers:

$$\begin{array}{r} 25 \text{ r } 8 \\ 34 \overline{)\,786} \\ \underline{6\ 8} \\ 178 \\ \underline{170} \\ 8 \end{array}$$

The teacher should see to it that children write their numbers large enough and keep them arranged in neat order.

20. *Mixing Problems of Different Types.* It is not advisable to have problems arranged so that they require the same fundamental operation for solution. Problems arranged in such a manner lead to mechanical, routine thinking. It is a better practice to mix up the arithmetic problems in a series so that they call for different fundamental operations in their solution.

Summary

In summary, it must be pointed out that not all of the foregoing suggested methods for improving problem solving in arithmetic are equally effective for all age and grade levels. The whole method of attacking a problem proves most effective with intermediate grade children and with children of average or better than average ability. Diagraming problems, making up problems, and dramatizing problems while effective at all grade levels usually appeal more to younger children and older children with average or below average ability. Estimating answers to problems and working problems using different methods cannot be employed until children have developed at least a fair degree of facility with numbers. Solving problems with irrelevant facts, finding missing facts in problems, and finishing incomplete problems are techniques that prove very helpful with children who are careless in the reading and analyzing of problems. The use of concrete aids in solving problems and checking answers to problems are most useful in the beginning stages of learning a new arithmetic process or topic. Lastly, working problems without numbers and orally solving problems are techniques that prove about equally effective for all age and grade levels, and it is highly recommended that a 10 or 15 minute daily period be reserved for this type of invaluable arithmetic instruction.

EDITOR'S NOTE. Dr. Koenker has summarized twenty fairly standard procedures in building ability in problem solving and he points out that pupils develop as individuals and that the method that seems opportune for one at a given time may not be the best method for his neighbor in the classroom. A teacher should have all of these procedures in readiness so that she can use one or more when the time seems appropriate. Our best generalization seems to be that pupils develop in problem solving ability when they are given problems to solve and receive the helps that the occasion seems to demand. The better pupils, particularly, should be encouraged to use more than one method as an aid in their depth of understanding. A generous amount of mental-oral work is advised because that is so frequently the mode required outside of school. It should also be remembered that the steps in mental-oral work often differ from the steps that would be used in the paper-pencil technique.

BOOK REVIEW

Arithmetoons #2, Lowry W. Harding. Dubuque, Iowa: William C. Brown Company, 1957. Paper, iii+99 pp., $1.50.

This is a small paper bound book of cartoons, most of which pertain to the teaching of arithmetic. According to the foreward, the cartoons are used to emphasize the main idea and elicit the personal involvement of the reader. Also, the purpose is to soften dislike and reduce fear of the subject as well as to stimulate interest. The person who would probably appreciate its contents most is the professional educator, of course, in the teaching of arithmetic. However, even the elementary arithmetic pupil might enjoy the ideas expressed in some of the cartoons. Many of them would be of value to the parent of the elementary school pupil. In fact, they are varied enough in their scope of ideas that any person interested in education might find that at least some of the treatments have an appeal for him.

This book contains forty-five cartoons. Opposite each cartoon is an explanation of the idea the cartoon exemplifies. Sometimes this is merely a statement of the idea and sometimes it is a quotation embellished by Harding. In some cases this explanation is helpful to the reader because the idea is not always readily discernible from the cartoon itself.

Although most of the subjects treated have to do with the teaching of arithmetic, some pertain to such items as thesis writing, the educators' use of time, teachers' and professors' salaries, evaluation and recommendations.

The use of cartoons to make a point is an excellent idea and should be more widely used by public school teachers as instructional material, bulletin board displays and in many other ways. Mr. Harding is to be commended for his contributions to this field. Any teacher might get ideas for such uses from this book.

INA MAE SILVEY

Divisibility and Prime Numbers

PAUL YEAROUT
Seattle, Washington

BEING an elementary discussion of some of the useful and interesting properties of numbers which are not often considered in elementary and secondary arithmetic courses.

Factorization and Primes

It is a peculiarity of some whole numbers that they can be obtained by multiplying together other whole numbers, such as $15=3\cdot5$, $35=5\cdot7$, $120=12\cdot10$. Such a number is called composite. The parts are called factors or divisors (two names so as not to get bored!). Thus 3 and 5 are factors of 15, 10 and 12 are divisors of 120. On the other hand, some numbers have only themselves and one as factors, like 5, 19, 31. These numbers are called prime numbers (or, simply, primes) and play an important part in the study of numbers. The first few primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29.

One of the first things to notice about a composite number is that it can always be factored so that all the factors are primes. In the examples above, 15 and 35 are already written this way. However, the factors 10 and 12 of 120 can each be factored further: $12=3\cdot4$, $10=2\cdot5$. If any of these new factors are still composite, they can be factored again. Thus, finally, $120=2\cdot2\cdot2\cdot3\cdot5$. This is usually written $120=2^3\cdot3\cdot5$. One of the basic facts of arithmetic is that no matter how a number is factored into its prime divisors, the results always the same. (Mathematicians call this statement the Fundamental Theorem of Arithmetic.) So, even if we start out $120=6\cdot20$, the final factorization will be the one found above.

Sometimes it is quite difficult to determine the prime factors of a given number. In fact, there are some extremely large numbers for which it is not known whether they are prime or composite, and others which are known to be composite but which have not yet been factored! It is possible simply to start out with the primes in increasing order and try them, but this can be tiresome and long, particularly when the given number is large. To help with this problem, Lehmer's *Factor Tables* give the factors of all numbers less than 10,000,000. There are also a few so-called divisibility rules which help to determine whether certain small numbers are factors or not:

Rule 1. Any number is divisible by 2, 4, or 8 if the last 1, 2, or 3 digits are divisible by 2, 4, or 8 respectively.

57942 has 2 as a factor, but not 4.
57932 has 4 as a factor, but not 8.
57832 has 8 as a factor.

[As a matter of fact, any number is divisible by 2^n in case the last n digits are divisible by 2^n. But, above $n=3$ ($2^3=8$) this rule is too much trouble to make it worthwhile.]

Rule 2. Any number is divisible by 3 or 9 if the sum of the digits is divisible by 3 or 9 respectively.

247362 is divisible by 3, but not 9, since

$$2+4+7+3+6+2=24$$

is divisible by 3, but not by 9.

74322 is divisible by 9, since

$$7+4+3+2+2=18$$

is divisible by 9.

Rule 3. Any number is divisible by 5 in case the last digit is 0 or 5.

Rule 4. Any number is divisible by 11 if the alternating sum of the digits is divisible by 11. By the alternating sum of the digits is meant the sum formed by putting a minus sign in front of the alternate digits. Thus,

the alternating sum of the digits of 827654 is

$$8-2+7-6+5-4=8,$$

and 7384938 is divisible by 11, since

$$7-3+8-4+9-3+8=22$$

is divisible by 11.

Also 2476386 is divisible by 11, since

$$2-4+7-6+3-8+6=0$$

is divisible by 11.

The only number less than twleve for which there is no simple rule is 7. For seven (and also thirteen) there are rules, but they are complicated and take almost as much time as the long division.

Example: Factor 107640.

1. First factor out any tens:

$$107640=10764\cdot 10.$$

2. Since 64 is divisible by 4, so is 10764.
3. Since

$$1+0+7+6+4=18$$

is divisible by 9, so is 10764.

4. Since

$$1-0+7-6+4=6$$

is not divisible by 11, neither is 10764.

We now know that 36 $(=4\cdot 9)$ is a factor of 10764. Divide it out:

$$10764=36\cdot 299.$$

299 is not divisible by 2, 3, 5, or 11. Try 7. It doesn't work. Try 13 (the next largest prime after 11). $299=13\cdot 23$. 23 is a prime. Hence

$$107640=10\cdot 36\cdot 13\cdot 23=2\cdot 5\cdot 4\cdot 9\cdot 13\cdot 23$$
$$=2^3\cdot 3^2\cdot 5\cdot 13\cdot 23.$$

Addition of Fractions

One useful application of prime factorization turns up in the process of finding the least common denominator of two or more fractions. For instance take the problem:

$$\frac{4}{15}+\frac{5}{24}+\frac{1}{36}.$$

The l.c.d. of these fractions can easily be found in this way:

1. Factor each denominator into its prime factors.

$$15=3\cdot 5,\quad 24=2^3\cdot 3,\quad 36=2^2\cdot 3^2.$$

2. The l.c.d. is the number built up by taking each prime which appears to the highest power (i.e. the most times) to which it appears in any one of the numbers.

$$\text{l.c.d.}=2^3\cdot 3^2\cdot 5=360,$$

since 2 appears to the third power in 24, 3 appears twice in 36 and 5 appears once in 15.

The next step in adding these fractions is to divide each denominator into the l.c.d. Again the prime factorization is useful. For example, to divide 360 by 15, simply remove the factors of 15 from the factorization of 360 and what is left is the quotient. After removing 3 and 5 from $2^3\cdot 3^2\cdot 5$ we have left $2^3\cdot 3=24$. That is to say, 360 divided by 15 is 24. Similarly for the other denominators. Then the addition is

$$\frac{4\cdot 24}{360}+\frac{5\cdot 15}{360}+\frac{10}{360}=\frac{96+75+10}{360}=\frac{181}{360}.$$

This last step can also be thought of as multiplying both numerator and denominator by the proper number so as to have on the bottom the common denominator. This "proper" number can be obtained as above from the prime factorizations.

Distributive Law

The distributive law is one of the rules of arithmetic which is so basic that it is frequently overlooked entirely. The distributive law states that in multiplying a product like $3(5+7)$ [which is usually done as $3\cdot 12=36$] the same result will be obtained if the multiplier outside the parenthesis is first multiplied into each of the terms inside and the results added:

$$3\cdot 5+3\cdot 7=15+21=36$$

i.e.

$$3(5+7)=3\cdot 5+3\cdot 7.$$

Here is one way in which the distributive law can be used to save effort in multiplication:

By noticing that $199=200-1$, write the product $46\cdot199$ as $46(200-1)$, and then, using the distributive law, this is

$$46\cdot200-46\cdot1=9200-46=9154.$$

These calculations can be done mentally, with a little practice, and the results simply written down. (Try $53\cdot298$.)

Here is another type problem:

$$26^2-26\cdot24$$

In this case, the distributive law is used in reverse. Since 26 appears in each term, the same result will be obtained if the problem is done

$$26(26-24)=26\cdot2=52.$$

This law can also be used in more complicated problems. In the problem

$$(9+14)(5+7)=23\cdot12=276,$$

remembering that an expression enclosed by by parentheses is a single quantity, even if written as a sum of several terms, we can use the distributive law *twice*, first thinking of $(9+14)$ as the "outside" multiplier, and write the above product as a sum

$$(9+14)(5)+(9+14)(7)$$

and, using the distributive law again on each of these,

$$9\cdot5+14\cdot5+9\cdot7+14\cdot7=45+70+63+98$$
$$=276.$$

In this case, there is probably more work involved in the second method than in the first. However, a special case of this rule is extremely interesting. Look at

$$(7-5)(7+5)=2\cdot12=24$$

in the above way:

$$(7-5)7+(7-5)5=7\cdot7-5\cdot7+7\cdot5-5\cdot5,$$

and notice that the sum of the two middle terms is zero, so that

$$(7-5)(7+5)=7^2-5^2.$$

Stated in words, the difference of the squares of two numbers is the difference of the numbers multiplied by their sum. Using this rule to multiply $29\cdot31$, notice that $29=30-1$ and $31=30+1$. Thus this multiplication becomes

$$29\cdot31=(30-1)(30+1)=900-1=899.$$

Likewise

$$13\cdot17=(15-2)(15+2)=225-4=221.$$

This rule can also be used in reverse with great ease and success:

$$29^2-24^2=(29+24)(29-24)=53\cdot5=265.$$

Such calculations soon become easy mental exercises, even for grade school students.

The introduction of these ideas (and others related to them) would have at least three advantages. First, an increase in calculational proficiency, second, an awakening of interest in arithmetic and, possibly, some appreciation of its stylistic beauty and, finally, the establishment of a broader background on which to base the teaching of algebra and thus make this traditionally difficult transition a little easier.

EDITOR'S NOTE. Mr. Yearout illustrates some interesting extensions of arithmetic that many younger teachers may not have encountered. In the standard texts of 100 years ago we found such topics as Prime Numbers and here we learned to add and subtract fractions after we have learned Greatest Common Divisor and Least Common Multiple. These topics have disappeared from most schools because the computational work is limited in terms of fractions that commonly occur in business and the trades and for these fractions the method of "inspection" coupled with a little reasoning suffices. However, we do have certain pupils with inquisitive minds and they ought to explore into the nature of work in arithmetic. These topics offer such an opportunity for gaining insight. Many people use the Distributive Law unknowingly when they perform an exercise such as $5\times\$2.98$ by thinking 10 cents less than \$15.00 or \$14.90. With the increased attention to putting more "mathematics" into the arithmetic for our more able pupils, we should welcome topics that provide an avenue for this type of study. Frequently, these pupils need a minimum of guidance but they should have ideas opened to them. As Mr. Yearout has said, the study of primes has real significance in more advanced mathematics.

Casting Out Nines and Other Numbers

LUCY E. DRISCOLL
Cook County Schools, Chicago

TODAY, PERHAPS MORE THAN ever before, we at least pay lip service to teaching for understanding in arithmetic. Too often, however, this "understanding" still implies mere knowledge of how to perform a mechanical process. Checking any one of the four processes of arithmetic by *casting out of nines* is a good illustration of this fact; though, occasionally some wide awake boy or girl in junior high school arithmetic challenges the teacher by asking such questions as: "Why does it work?" "Will it work with any other number?" These are good, intelligent questions, but questions which many teachers who feel rather secure in teaching most of the arithmetic find difficulty in answering. And yet, at least partial answers are understandable, interesting and challenging to pupils.

"Why does it work?" The full explanation requires mathematical equations beyond the grasp of the students in the elementary school. But let's examine the interesting aspects which they can understand, some of which they can be helped to discover for themselves.

When we cast out the nines from a number we generally do it by finding the sum of the digits, and then doing one of two things: (1) dividing the total by nine and using the residual, or (2) again adding the digits until we have a one place number, which if it should be 9 we "cast out" leaving no residual. For example: in casting out nines from 267,345, we find the sum of the digits to be 27, which if we divide by nine leaves no remainder, or, adding the 2 and 7 gives 9 which we "cast out" leaving no residual. But, if we change the number to 367,345 and perform the above operations we find we have a "1" as the residual. But, "Why does this work with 9 and not with 7 or some other number?" some youngster will be sure to ask. And the answer to this opens up a very interesting discussion. In every multiple of nine, the sum of the digits is 9 or a multiple of 9; hence, the rule for testing any number to find whether or not it is evenly divisible by 9. Many teachers call attention to the fact that in the "table" of 9's the sum of the digits in the product is 9; but, they fail to indicate that the same is true for every multiple of 9. The only other one place number for which this is true is 3; the sum of the digits of any multiple of 3 is evenly divisible by 3, and this, too, provides us with another practical test for divisibility.

Does it follow then, that we can check any process by casting out of 3's in the same way we cast out 9's? Yes, it does, and pupils are interested in being led to this discovery. In fact, casting out 3's has some advantages over casting out 9's; for example, the residual, if any, will be only 1 or 2.

Now, why does the casting out of 9's and 3's work, but not 7? Simply because 9 and 3 are the only integers which follow the principle that the sum of the digits of their multiples is divisible by the number.

However, this leads us to still other interesting "discoveries." The residual number we have after casting out 9 or 3 is the same number we would have as the remainder if we divided the original number by 9 or 3. And, if we chose to use any other number as a *divisor*, the remainders could be used for the purpose of checking just as we use the residual numbers in casting out 9's or 3's.

Let's take an example for each process and check by casting out different numbers by the division process itself. Pupils will enjoy experimenting with this and finding that "just any number" can be used in this way, as a means of checking.

Addition:	7326
(checked by	9451
casting out 7)	3267
	1583
	21627

Dividing each addend by 7, we have remainders of 4, 1, 5 and 1, which totals 11. Dividing 11 by 7 we have 1, and a remainder of 4. Dividing the sum 21,627, by 7 we have 3,089 and also a remainder of 4. It checks.

Subtraction:	21,634
(checked by	18,523
casting out 4)	3,111

Dividing the minuend, 21,634, by 4, we have 5,408 and a remainder of 2. The subtrahend, 18,523 divided by 4 is 4,630 and a remainder of 3. Since, in arithmetic, we do not subtract 3 from 2, we add a "4" to the 2 and then subtract 3 from 6, giving as a remainder of 3. Dividing the difference, 3,111 by 4 we have 777 and also a remainder of 3. It "checked."

Multiplication:	2569
(checked by casting	75
out 8)	12845
	17983
	192675

Dividing the multiplicand, 2,569 by 8, we have 321 and a remainder of 1. Dividing the multiplier 75 by 8 we have 9 and a remainder of 3. Multiplying these remainders -1×3, we have 3. Dividing the product, 192,675, by 8 we have 24,084 and also the remainder 3. It "checked."

	236
Division:	251) 59346
(check by	502
casting out 11)	914
	753
	1616
	1506
	110

Checking by using 11 as the divisor:

The quotient: $236\div11$ is 21—remainder 5

The divisor: $251\div11$ is 22—remainder 9

The dividend: $59346\div11$ is 5,395—remainder 1

The remainder $110\div11$ is 10—remainder 0

Using these residuals for checking—$(5\times9)+0=45$

$45\div11=4$ and remainder 1, which "agrees" with the 1 which is the residual in the dividend. It "checked."

No attempt has been made here to explain the process of casting out 9's. That process has been adequately discussed in previous articles in THE ARITHMETIC TEACHER and in books on the teaching of arithmetic. The object here has been merely to clarify the principles involved by explaining what is really happening when we cast out 9's and showing how these principles can be extended to apply to checking with other numbers as well. Pupils and teachers alike should find that this adds a bit of "spice" to this old and well known process.

EDITOR'S NOTE. Miss Driscoll urges that we not only use checks of nines but also other numbers and that in doing so we give a breadth of interest and understanding to our work with pupils. The actual proofs of these checks usually is done by applying the principle of the "remainder theorem" of algebra. But it is not necessary for pupils to understand fully why something "works," it is satisfying to them to know that it does work. Usually some pupils will find examples in which these checks do not reveal an incorrect answer. If they do this with contemplation, they are at the same time developing a further depth of understanding of the nature of numbers in our system. In THE ARITHMETIC TEACHER for November, 1956 Miss Moore has given a mathematical proof of the check by nines. The same reasoning can be applied to other similar checks.

Improving the Mathematical Competency of Teachers in Training

WILBUR WAGGONER
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ONE NEED NOT READ too widely in the educational literature of today to note a concern for the mathematical capabilities of the persons who are preparing for positions as elementary school arithmetic teachers. "Less than 20 per cent of prospective elementary school teachers understand arithmetic. One study of 211 teachers showed that 150 of them had an abiding hatred for it."[1] The above quotation appeared in *Parade*, a magazine written for the lay public. A further discussion on the problem of competency in subject-matter of arithmetic teachers may be found in *Problems of Mathematical Education*[2] published by Educational Testing Service. This report says, "In majority of cases an individual with ambition to teach in an elementary school can matriculate without showing any high school mathematics on his record. He can graduate without studying college mathematics."

The latter statement is not true at Central Michigan College. Candidates for a certificate to teach in an elementary school must successfully complete MATHEMATICS 151, Remedial and Functional Mathematics. This course is designed to alleviate any weaknesses in the fundamental processes of arithmetic, to provide the necessary mathematical tools for the work of the elementary curriculum, and to acquaint the students with some of the social and economic applications of mathematics.

MATHEMATICS 151 is a three-semester-hour credit course which meets five days per week. No credit is given the student for the remedial section of the course which is a redevelopment of skill in computation the student should have previously acquired. All elementary education students must take MATHEMATICS 151 regardless of demonstration of prior competency on a mathematical achievement test. However, as soon as a student completes the remedial part of the course, he no longer attends class the two days of the week devoted to remedial work.

Areas of Remedial Work

The remedial section of MATHEMATICS 151 consists of eleven tests that the student must pass to acquire credit in the course. The eleven remedial tests ate: (1) addition of whole numbers, (2) subtraction of whole numbers, (3) multiplication of whole numbers, (4) division of whole numbers, (5) common fractions, (6) decimal fractions, (7) percentage, (8) denominate numbers, (9) ratio, proportion, and averages, (10) geometric measurement, and (11) reading and writing numbers. The first four tests are timed tests. Two days a week are devoted to preparation for and taking these tests. A student may take each test as often as is necessary to achieve the desired competency.

Developmental Work

The functional aspect of the course is designed to aid the student in understanding the nature of arithmetic so that as a teacher he may make arithmetic a meaningful, understandable subject rather than a series of memorized algorithms. The functional section of the course consists of six units. These units are: (1) the history of mathematics and our number system, (2) roots and powers, (3) measurement, (4) graphs, (5) statistics, and (6) applications.

[1] Ed Kiester, "Why Johnny Can't Add," *Parade*, pp. 12–17, October 7, 1956.

[2] Henry S. Dyer, Robert Kalin, and Frederick M. Lord, *Problems in Mathematics Education*, Princeton, New Jersey: Educational Testing Service, 1956, pp. 12–17.

The first unit acquaints the student with the history of mathematics and ancient number systems such as the Egyptian, Babylonian, Greek, Mayan, and Roman. Various mechanical aids to computation are studied from earliest counters to modern digital computers. The use of the abacus and place value charts as instructional aids is given considerable attention. Various number bases such as binary, decimal and, duodecimal are considered so that a student may understand the meaning of base and place value. Students in MATHEMATICS 151 are expected to be able to do fundamental operations with any number as a base.

The second unit introduces the student to factors, the meaning of exponents and powers, and to the use of formula, approximation, interpolation, and factoring in finding roots. In the unit on measurement, the topics considered are the history of measurement, accuracy and precision in measurement, mensuration formulas, the metric system, Pythagorean Theorem, and denominate numbers. The students receive a rigorous treatment of the concept of approximation in measurement. This unit covers measurement from man's first attempt to standardize units of measurement to the use of gage blocks and interferometry in modern industry.

The various types of graphs, scale drawings and their applications to education, business, and general education are studied in unit four. The fifth unit teaches the student how to compute measures of central tendency, measures of dispersion, coefficients of correlation and index numbers. The final unit considers the application of mathematics to such topics as budgets, insurance, investments, taxes, installment buying, and interest.

Throughout the functional section of MATHEMATICS 151, the emphasis is on the fact that mathematics is a logical, sequential science which can be taught as an enjoyable, learnable subject. Although Remedial and Functional Mathematics is not a methods course, it is not possible to completely divorce a content course such as this from methodology. From time to time suggestions as to how arithmetic may be taught with meaning are given.

Does such a course as described above improve the mathematical skills of prospective teachers of elementary arithmetic? To help answer this question the "Davis Test of Functional Competence in Mathematics"[3] was administered at the beginning and end of the course by four different instructors in five different sections of the class. This test of eighty items is based on the essentials for functional competence in mathematics as outlined by the Commission for Post War Plans of the National Council of Teachers of Mathematics. The results indicate "the extent to which those mathematical skills and abilities considered necessary for effective living in a modern world, have been acquired."[4] Scaled scores of achievement are interpreted by percentile norms for the total school population at the mid year and the end of the year for grades nine through twelve.

One hundred thirty-two students took both tests. The mean increase in number of items correct was 7.7 items. This increase is significantly different from zero at the .01 level of confidence. When the hypothesis was tested that the increase in number of items correct was at least six items, this hypothesis could not be rejected at the .01 level of confidence. Probability limits were computed on the population parameter representing increase in the number of items answered correctly. Using a probability of 99 per cent, the probability limits were from 6.3 to 9.0. This implies that the true mean increase in 99 cases out of 100 would be straddled by these two scores.

The mean score at the beginning of the course was at the fiftieth percentile point on the end of the year norms for grade twelve. The mean score at the end of the semester was at the seventy-first percentile point on the same norms.

[3] David J. Davis, *Davis Test of Functional Competence in Mathematics*. Chicago: World Book Co., 1951.
[4] *Ibid.*, p. 7.

It is believed that the course described will help to provide elementary arithmetic teachers who are not afraid of arithmetic, and who will have the requisite mathematical skills to use their knowledge of methods of teaching arithmetic to teach children to understand, enjoy, appreciate, and use arithmetic. From the statistical analysis of the test scores, it is certain that the teachers themselves will have a greater competency in using and understanding mathematics than they possessed prior to taking Remedial and Functional Mathematics.

EDITOR'S NOTE. The picture of training in arithmetic and other areas of mathematics for prospective elementary school teachers is not all black in the United States. There are many training colleges that are doing a fine job in two important aspects of the problem. These are: (a) building a background of understanding and of performance in arithmetic and (b) developing a more advanced understanding of important areas of mathematics that are worthwhile to the student as an educated citizen and teacher. Two circumstances are troublesome: (a) the college training program that presumes that one who has studied child development will be able to teach anything expected in an elementary school and (b) former teachers who are returning to the profession and who had a meager normal-school training some years ago. Of course, a knowledge of child growth and development is important to the teacher of arithmetic just as a knowledge of physiology is important to a surgeon but most of us would like to have our surgery performed by one who has additional training specifically in surgery

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Offerings and Enrollments in Science and Mathematics

Pamphlet Number 120

U. S. DEPT. OF HEALTH, EDUCATION, AND WELFARE

This pamphlet of 44 pages was prepared by Kenneth Brown, Specialist for Mathematics, and Ellsworth Osbourn, Specialist for Science of the U. S. Office of Education. Mathematics teachers should have a copy for reference. It is filled with charts and tables showing enrollments through 1956. The following facts are summarized in the pamphlet.

1. Between 1954 and 1956 the percentage of 11th grade pupils taking chemistry and 12th grade pupils taking physics increased.
2. The number and percentage of pupils enrolled in college preparatory mathematics increased from 1954 to 1956.
3. Between 1900 and 1956 the percentage of total high school students taking physics declined from 19 to 4.4, but during the same period the number increased from 98,846 to 309,600.
4. In 1956, of all pupils enrolled in the 12th grade of public high schools, 95.2 per cent could have taken physics or chemistry. In other words, only 4.8 per cent did not have access to such a course.
5. Two-fifths of the high school pupils took plane geometry, a course usually required for college entrance.
6. One hundred thousand high school seniors were in schools offering no advanced high school mathematics.
7. Ninety per cent of the 10th grade pupils in the South Atlantic region took biology, but only 64.7 in the Pacific Coast region.
8. The percentage of pupils enrolled in certain mathematics courses in one region was five times the percentage in another region.
9. Four times as many boys as girls took 12th grade mathematics.
10. The number of schools offering neither physics nor chemistry has declined.

The Neglected Role of the Decimal Point

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THE MODERN APPROACH to elementary arithmetic places considerable emphasis upon the basic structure of our Hindu-Arabic system of number notation. It is felt that upon such a foundation the algorisms—those rules of procedure by which we combine, separate and compare numbers by working with the numerals that represent these numbers—can best be understood.* It is almost trivial to state that understanding, not rote manipulation, is the keystone of this modern approach.

In the early grades great care is taken to acquaint the student with the fact that our system of whole number notation employs ten basic digits, used singly to represent numbers from zero to nine or compounded into multidigit numerals to represent numbers beyond nine. Stress is placed on the vital concept of place-value, so necessary in determining the number value for which a specific numeral stands. Special note is made of the constant 1 to 10 ratio which exists between adjoining places in this system of notation: 1 in the tens' place is the equivalent of ten of the ones; 1 in the hundreds' place is the equivalent of ten of the tens, and so on.

But when the time comes to extend this system of number notation into the "decimal fraction" realm, we do not generally do so thorough a job. Because of the student's previous knowledge of fractions as parts of wholes, coupled with his awareness of the 1 to 10 place value ratio, he usually accepts readily the notion of this extension to the right. Before this adventure has progressed very far, the all-important decimal point is necessarily introduced. In all likelihood the teacher, supported by her textbook, will say something like: "Now we are going to need a mark of some kind to *separate* the whole number part of this new number expression from the fraction part. In this country we use a dot."

This is grossly misleading. In our system of number notation there really exists no fundamental or systematic need for a separator; but there is a fundamental and systematic need for some kind of *place-value designator.*

When the Hindu-Arabic system of number notation is restricted to whole-number expressions, there operates a tacitly accepted convention for determining the value of a numeral: the digit at the far right is assumed to be the one's digit (unless otherwise specified, as in "62 tens"). However, when the scheme is extended to right of the ones' place, resulting numerals no longer have a readily identifiable "low man on the totem pole" by which the observer may get his numerical bearings. So something must be done.

This need for a designator was recognized right from the beginning, from the time that Simon Stevin first advanced the idea of extending the system in his *La Disme* in 1585. Though Stevin's idea was readily grasped, his suggested notation was not, and a wide variety of practices ensued. Eventually the dot prevailed, though not universally. (Even today in certain countries of Europe, for instance, a comma is used; in England the

* (AUTHOR'S NOTE: Although the numeral-number distinction used in this article is rapidly gaining acceptance, it is by no means universally used or understood. So by way of definition, at least for the purposes of this article: number means the number *concept;* a numeral (or number expression) is that symbol or set of symbols one writes down to represent the number or number concept; numerals may be either single-digit (as 5, 3, 7) or multidigit (as 42, 803, 5,729); the ten symbols (0, 1, 2 . . . . . 9) that we use either singly or in combinations to express numerals are called digits.)

dot is used at a different level than is the practice in this country.)

The important function of this designator, then, is not to separate the numeral into two parts but rather to single out a certain digit in the numeral as having some predecided place-value. And, of course, this key digit need not be the ones' digit. For example, if somehow you know that the digit 8 in the numeral 382574 is the hundreds' digit, then the place values of each of the other digits is immediately known. You know, therefore, that the numeral stands for 3 thousands+8 *hundreds*+2 tens +5 ones +7 tenths +4 hundredths. Nor is it important in this regard that the designator occur *between* two digits: a dot *over* or *under* the key digit would be just as effective.

It is often argued that *because* the decimal point as we use it separates the whole and the fractional part of the number expression, it provides a method of identifying a specific digit as having a specific place-value (i.e., the ones' digit is the digit immediately to the *left* of the decimal point). To me this reflects a confusion of cause and effect. It would be much more accurate to say that *because* the place-value designator is stationed to the right of the ones' digit it *in effect* separates the whole and the fractional part of the numeral.

Whether the reader thinks this argument has validity or is simply much ado about nothing, few will deny that a singular emphasis on the separator role of the decimal point is hazardous to a full understanding of the Hindu-Arabic system of number notation. Unquestionably, a concomitant of this singular emphasis is the tendency for the decimal point to become a focal point—the center of balance, as it were—of the number expression. With the decimal point in this role, our complete system of number notation (endless places to the left and right) appears to be somewhat askew: the place immediately to the left of the decimal point is *ones*, the place immediately to the right of the decimal point is *tenths;* the second places to the left and right of the decimal point are *tens* and *hundredths*, respectively; the third places to the left and right of the decimal point are *hundreds* and *thousandths*, respectively; and so on. I have heard both teachers and students, having observed this fact, remark, "Too bad, isn't it, that the system isn't better balanced?"

But the system *is* balanced. It is beautifully symmetric, but *with respect to the ones' place*, not the decimal point: the places immediately to the left and right of the ones place are *tens* and *tenths*, respectively; the second places to the left and right of the ones place are *hundreds* and *hundredths*, respectively; the third places to the left and right of the ones place are *thousands* and *thousandths*, respectively; and so on. In short, the focal point must be the ones place, not the decimal point, if the symmetry of the system is to be perceived.

Consequently, if this hazard is to be avoided and the student is to become fully aware of the great internal consistency of the full Hindu-Arabic system of number notation, then the designator role of the decimal point must be fully understood.

EDITOR'S NOTE. Dr. Mueller asks that we center our attention on the ones' place as the symmetrical balance in our writing of numbers. This can be done in teaching. It would give more significance to the names of whole numbers and decimal fractions. The decimal point or some similar device is needed but confusing conclusions are frequently drawn because the point seemingly occupies "space" in the array of digits. It is not uncommon to hear people argue that tenths are 3 places removed from tens. We apparently need to proceed from the use of a separator as in the case of money to a more sophisticated level in decimal fractions if we are to attain a high level of understanding.

Meaning for Multiplication of Fractions

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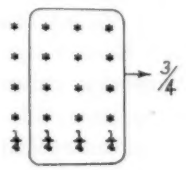
"NECESSITY IS THE MOTHER of invention," so says an old adage. Thus it isn't until there is a real need to do something that most of us ever get around to doing it. It is the teacher who teaches multiplication of fractions who really gets down to finding the meaning of it when she must present it to her middle-grade class. Even so the children never find the meaning until they see the need. The need is self-explanatory with the meaning. Since they go hand in hand, we frequently omit both to present the method alone.

To take $\frac{3}{4}$ of 16 we draw 16 objects on the blackboard and mark off $\frac{3}{4}$ of them:



1/4 of 16 = 4

3/4 of 16 = 3×1/4 of 16

= 3×4 = 12

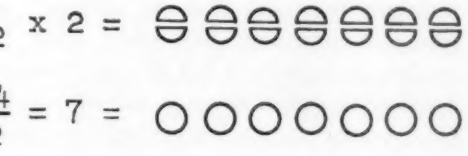
However to multiply $\frac{3}{4}$ by 16 we take $\frac{3}{4}$ of 16 times. If necessary (and sometimes it is, if a child doesn't understand) we may draw 16 $\frac{3}{4}$-size pieces on paper, cut them out and put them together. $\frac{3}{4}\times 16 = 48/4 = 12$.

To multiply 16 by $\frac{3}{4}$ we are taking only $\frac{3}{4}$ of each of the 16 wholes, which is merely a reverse statement of the above.

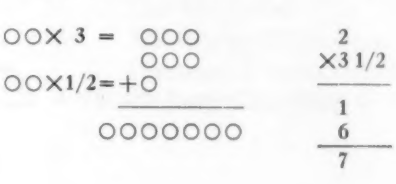
16×3/4 = 48/4 = 12

Any of these may be interchanged in finding results but the meaning is different. The child may wonder why he must learn three methods when one would seemingly suffice. Thus presentation of the meaning becomes most important. A problem for each phase may help to illustrate and point out the need for learning the different ways of working these.

In order to multiply a mixed number by a whole number we took out our previously cut sets of whole circles and halves.



When we had 2 wholes which we wanted to multiply by $3\frac{1}{2}$ we showed it in this form:



The multiplication of a fraction by a fraction can be shown on the flannelboard. When we multiply a fraction by a fraction we have a smaller piece, even as we had a smaller number when we multiplied a whole number by a fraction, as in the above.

To show $\frac{1}{2}$ of $\frac{1}{4}$, take a one-fourth fraction of a circle and separate it into halves. Each of these halves (of the fourth) makes one-eighth part of the circle. Thus $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$.

To find $\frac{1}{2}$ of $\frac{3}{4}$, take $\frac{3}{4}$ of a circle and divide it into two equal pieces. Lay on the $\frac{1}{8}$-size pieces and note that $\frac{1}{2}$ of $\frac{3}{4} = \frac{3}{8}$.

Can you show that $\frac{3}{4}$ of $\frac{1}{2} = \frac{3}{8}$ by dividing the half into fourths and then taking 3 of these pieces?

This explanation may precede the teaching of the multiplication of a mixed number times a mixed number. For in this we change both numbers to fractions, even though improper fractions. It is not necessary as a rule, in beginning this study with fifth graders to explain in full the mechanism of a mixed number times a mixed number. However for the teacher's benefit we present the following:

$$2\tfrac{1}{2} \times 2\tfrac{1}{2} = 5/2 \times 5/2 = 25 \times \tfrac{1}{2} \text{ of } \tfrac{1}{2}$$

EDITOR'S NOTE. Sometimes we confuse youngsters by our two common methods of expressing multiplication, i.e. "times" and "Multiplied by." In general, pupils seems to grasp the idea of "times" easier than they do the older Latin expression. Mrs. Hoffman points to an important idea in teaching and that is the relationship of *need* and *meaning*. This may be taken into another avenue as the "need for meaning" on the part of pupils. It seems so sensible to illustrate the meaning of work with fractions and this should begin with the simplest concepts and extend to the stage when abstract work in computations has significance. One must be careful lest certain rather fanciful representations become more confusing and serve as a deterrent to good learning. All teachers have experienced the thrill that comes to a child (and to the teacher) when a youngster first really sees that multiplication involving a proper fraction results in an answer that is smaller than the other factor in the multiplication. Perhaps too many pupils in lower grades were told that "you always get more when you multiply."

The Pledge of an Arithmetic Teacher

I will give my pupils time to think and an opportunity to discuss their considered thoughts, even though they might make mistakes. I know that children need practice in thinking as much as they need practice in other numerical skills.

I will use care in selecting teaching aids so that understanding and effective learning can result from their use.

I will organize the block of time in which I teach arithmetic to provide some experience in thinking and in problem solving for every child in my classroom, from the pupil with the most immature numerical concepts to the most gifted student.

I will continually endeavor to ask the kinds of questions and give the kinds of directions which will encourage my pupils to think.

I will think positively about my class and never make such negative statements as, "My class can't do written problems." I know that full understanding does not arrive in the mind of every child at the same time nor in the same way. I must therefore remember that even though every teacher before we might have done a superb job of teaching, the boys and girls in my class may vary greatly in their individual levels of learning and understanding. My share of responsibility in the teaching cycle is to find out what my class can do and what I can do to help the class.

I will try to focus my teaching for effective learning. I will therefore need to help my pupils locate their areas of numerical strengths and numerical needs. Their needs will continually guide my planning of what and how to teach.

I will use special care in teaching and in maintaining the meaning of numbers and the meaning of the four fundamental processes. Pupils grow in their ability to perceive relationships and interrelationships and can profit from guidance at their respective levels of expanding development.

I will observe sound principles of learning in using drill. I will first find out what a pupil thoroughly understands. When he needs help in thinking relationships, I will interpret such evidence that drill would be unprofitable. When relationships are clearly understood, I will encourage both immature and able pupils to use practice to help them maintain and to help them become proficient in the use of those facts each pupil understands.

I must remember that I cannot teach today's lesson on the basis that every pupil will remember and fully understand all that we had yesterday, all that we had a week ago, all that the class had last year. I may often have to build background to help my class bridge the gap between what they thoroughly comprehend from previous experience and the new concepts they are striving to understand.

I will plan my year's work in such a way that I can feel that I have had time to teach arithmetic. This means that I must use judgment in making assignments and must distribute the use of my time between group and individual needs to encourage the successful growth of each pupil.

I will encourage my pupils to be resourceful in problem solving and remember that the methods of thinking as well as the solution are important. I will also show appreciation for evidences of fine thinking and will try to provide an atmosphere in which the students have an opportunity to cultivate genuine respect for their own and other pupils' inspired solutions.

I must remember to guide my students into using arithmetic whenever strategic opportunities occur in such areas as social studies, science and the other activities of their school day. I must also remember that the content of the same areas and activities may offer some valuable sources of problem-solving material for use during the arithmetic period.

I will share my enthusiasm and appreciation for the value of elementary mathematics with my pupils. Since they are living in an increasingly more scientific world, I must accept some of the responsibility for helping them catch the vision of the many avenues of possibility which surround them, so that they can more fully benefit from and more fully contribute to the thriving cultural and economic forces of their time.

Contributed by
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"I Like Math Because . . ."

Bob Cross*
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The main reason I like Math is that it has always elevated my opinion of myself. Every time I successfully work out a problem, I have finished one more little job that lies obscured in the countless small tasks that mark our everyday life. It is said that to complete a job is one thing; then to do it well is another. True, but there is only one way to complete a Math problem, and that is to find the right answer. Until then the task is incomplete and will remain incomplete as a nagging marker in one's brain.

There is only one solution to every problem, no matter how many parts it may have, or how many ways to find it. For this reason I try to keep myself from saying, "Well, it could be *this* or *that*." The one solution is the right solution, and, most of the time, there are many, many ways to find it. Some are shorter and some are longer, but I try not to let this bother me. Short cuts are fine, but for the best results, I like to work a problem all ways possible. Many times, of course, time does not permit this, but when one is working a problem for his own satisfaction, I'm sure he'll gain mathematical confidence by doing it both the long and short ways.

One of the most wonderful things about mathematics is the tremendous number of fields to which it is applied. One simply cannot enter any sort of a respectable profession without a well balanced knowledge of this science. If I were to attempt to name the fields which require a knowledge of some math, the remainder of this page could not approach holding one thousandth of the data. Surely no one can in any way think that he will be able to "make it" in life without the basic understanding of math that is *so* necessary even to a housewife. I appreciate this science because I know it will help me whenever I call on it. I probably should have said that math would help me to help myself, because surely math, better than anything else, helps one to make his mind work for him.

There is an art to learning math, just as there is an art to anything one undertakes, and this art is simply, "climbing the ladder," as I call it. One must grasp each rung on a ladder in order to climb safely to the top. In math, too, we must grasp each rung in the ladder of learning. We must keep our minds open and take each new thing for what it's worth *now!* We shouldn't dwell in the past or future (especially the future). I think that more people fail this subject because they can't progress with the majority than for any other reason. Too many people turn to the back of the text book and worry about the coming lessons instead of building a firm foundation now and adding the roof later. You just can't put the roof in mid air and expect it to stay there. The key is just to keep up day by day, bring in the assignment, and do the best you can to understand the present situation. In this way you can build an interest that will be invaluable for the rest of your life.

You know, I've found that the science of mathematics is a subject that gives a challenge to one's willpower. If one can keep himself from letting other things interfere with his concentration and can make himself pay attention and "climb the ladder" carefully, then he has control over himself, and having self control is about the finest achievement attainable. Just feeling a certain amount of confidence and willpower can cultivate a brilliant mind and I can think of no better place for this than a mathematics class.

* Bob Cross is a student in high school. He prepared this paper under the guidance of his teacher, Miss Blanche Mathison.

There is one thing I've found that many, many students need to know, and that thing is the importance of a teacher. A teacher is someone that really cares about what happens to you and me. They are the only key to understanding any subject, and I can't find a single one where a teacher is more needed than in math. I always find it much easier when I cooperate with my teachers in everything.

Whenever I encounter a hard problem in *all* subjects I find it very wise to call on good old math for assistance. Yes, even in our daily problems we can use a general knowledge of mathematics and find the right solution. Take, for example, the family budget, banking problems, and other common problems that we encounter from day to day. Surely our balanced knowledge of mathematics pays off bountifully in the solving of these problems. Anything can be accomplished through good deductive reasoning which we learn from a study of Geometry. This was proved many years ago by Sir Francis Bacon who said that the only true knowledge is that which is proved by deduction.

I think a mathematical attitude is the most important accomplishment of math courses. If one can just learn to work things out, step by step carefully, he can progress without hesitation. Think of the aviator, for instance. Here, as much as ever, a mathematical attitude will prove invaluable. The pilot of a plane must think in terms of a math problem. If each step and action is carried out correctly, then the right results are assured. Through mathematics, some things can be perfect, obstructed only by God himself, who, I'm sure, doesn't choose to break a law made by himself. In order to get things done, we must think, act, and live mathematically.

This does not mean be drab, for math is anything but drab. Oh no, indeed, this is a subject of unbounded interest and unseen horizons yet to be reached, and when those are reached, there will be others and after those, others, and so on through eternity. I'm sure anyone with a good mind of any sort can develop a lasting interest in mathematics. Think of the laws of mathematical nature involved in the solar system and all parts of our vast universe. Think of how the earth, a mass of unliving material travels its perfect path around the Sun without a flaw, and maybe some people don't know it, but the Sun, too, is on a perfect course, a course that in a great many years will take us near the now bright star Vega. Although we travel on that course at a tremendous rate of speed, the distance yet to go is so great that Vega may not be bright, or even exist by the time we reach its vicinity. If the distance we travel each year were compared to a period on this page, we would still have over a quarter of a mile to go. Yes, math is a tremendous collossus but all things abide by its laws, just as they did trillions and octillions of years ago and just as they will always.

It's not hard to love math. It's not hard to excel in math. It's not truly hard to understand math, and, yet, it's not all that easy, but, oh, it's so hard to to live without it. Nearly all our modern conveniences and pleasures are involved in mathematics since the men who built them were scientists who had to have a thorough knowledge of the laws of math to do what they have done. Think of the laws of gravity, the universe, even oblivion, are within the understandable realms of this wonderful science.

My advice to everyone is to accept math and cultivate an interest in it.

EDITOR'S NOTE. Young Bob Cross has expressed himself very well. Note that he not only points out the universality of mathematics but he also gives some advice on problem solving when he suggests solving problems by several methods. He points to the elements of certainty and discipline in the study of mathematics. One suspects that here is a young man who has the capacity for and the will to achieve the goals that he has set for himself. Our best wishes to him.

How Profitable Is the Usual Problem Work in Arithmetic?

GUY M. WILSON
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THERE are at least three distinct types of material in arithmetic,—drill, problem, and appreciation. Each type of material carries a distinct aim.

The aim in drill is perfect mastery of the processes which make up 90% of ordinary adult figuring. This means the four fundamental processes and little more. Drill is usually begun too soon and carried too far; it should be kept well behind meaning and motivation. The load must be small enough to make mastery possible.

Much of the appreciation material of arithmetic is for reference only, for most children. It can include complicated fractions, decimals beyond a reading knowledge, ratio, mensuration, and short units in algebra and geometry. However, for the bright pupil who likes mathematics, this little-used material can become more than topics for reference; it can become his intellectual recreational opportunity, and an introduction to phases of mathematics which he will pursue further in high school and college.

The true aim of problem work is the development of judgment in business. The explanation of this aim is the main purpose of this brief article.

Arithmetic is a useful tool for checking on alternate business decisions. Which is the better plan? Which is the less expensive? Which promises the greater return? Should I pay the extra price demanded for a vacant lot in a well-established neighborhood? Should I buy a new car or a used car? These women's dresses, says a women's apparel shop-owner, are not moving as expected; shall I mark them down and insert a display ad in the Sunday papers? If I do, can I still make a profit?

And always the business man or woman figures in a situation with which he is familiar. A druggist in Boston, with no business experience outside the Boston area, isn't spending his time figuring on farm crops in Ohio or Minnesota.

Furthermore, in real life figuring is always in terms of accumulated data. A department store manager studies his sales and profits reports to see which departments are doing well, which if any are losing money for the store. Which departments need more help? Where can expenses be reduced? And in all of this the manager is figuring in a field of experience, where he knows the meaning of what he is doing.

By contrast, as I look into the pages of a sixth grade arithmetic text, I see that a sixth grade pupil is expected to figure on beehives and honey, cartons of eggs, buying farm land, selling butcher cattle, buying wire fencing to fence a field and at a price per rod that is far from realistic, and on and on indefinitely. One text series analyzed by a graduate student, shows over 2000 different business situations for pupils to figure. Study of adult figuring shows that the average adult seldom figures on as many as fifty different business situations.

Without an experience basis, written problems are meaningless; they are merely disguised drill. If an adult or a business man seldom or never figures in situations outside his experience, why should pupils in school be asked to do so?

The seven "problems" on one page of the aforementioned sixth grade text indicate:

(1) that Mrs. Carter bought 5 clotheslines, each 50 ft. long. What would Mrs. Carter do with *five* clotheslines?

(2) that Helen bought a dress for $10.95; *probable*, but what about quality, etc. Did she use good judgment?

(3) that a rancher with 1500 sheep, sheared half of them and had an average of so many pounds of wool per fleece. How many pupils or teachers would know how much a fleece should weigh? Only about one-sixth of our population now live on farms, and most farms do not keep sheep; so only a small fraction of pupils would have any experience basis for judgment. And why would a rancher shear only half his sheep?

(4) that Mr. Ross has 5 horses (as well as cattle and sheep) on his 160-acre farm. Horses are now improbable on a farm of that size; the tractor has taken over.

(5) that bus fares are 7¢; *improbable.*

(6) that Mrs. Ward bought ground round steak at 52¢ a pound.

(7) that beaver skins were sold at $7.50 each. How many pupils (or teachers) have had any experience with the sale of beaver skins?

This page of written, or verbal, problems leaves a critical reader with many questions. Can school time be profitably spent on such work, especially in view of the fact that functional problem units can provide a business approach and an opportunity for judgment as in real life?

It is true that textbook "problems" work is not now as bad as it was a few decades ago. The "catch" problem has been largely eliminated, and "unified situations" replace in a measure, the strictly isolated problem lists. But we need to go much further. Functional problem units, well chosen and properly adapted, can provide opportunities for judgment on a basis of collected data, fairly comparable to the judgment opportunities in real life. There is now adequate proof that this is a feasible undertaking and that superior values accrue to the pupils. See reference No. 7. It is not to our credit that Colburn's idea of "number by development," or mental discipline, continues to direct much of our work in arithmetic, particularly our problem work.

In real life, arithmetic is the handmaid of business, and in our schools arithmetic, particularly the problem phases of arithmetic, need not be artificial and meaningless; problems can have a significant business basis. The boy who went to the store with a dime and came back with a Hershey bar and a great big nickel, had experienced arithmetic on a real life basis; he had had a business experience. Jim had learned to make his own bed during a few weeks at a summer camp. His wise mother capitalized this experience by hiring him to take over bed making in the home, and this resulted in a growing savings account.

All the problem work in arithmetic can be developed as business experience. Most units will be based upon home experiences and the family budget, but some pupils will have the necessary experience to permit the development of vocational problem units.

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EDITOR'S NOTE. Whether or not we agree with Professor Wilson we must acknowledge his challenge to reappraise our view of the role of the written problem in arithmetic. Is he correct in saying, "The true aim of problem work is the development of judgment in business"? Is the primary aim of a problem dealing with livestock to give a knowledge of the business of dealing with livestock? Are we to restrict problems in textbooks to those that are current and common to the age-grade level of pupils who use the particular text? Are we expecting textbooks for the Boston area and different books for the farms of Dakota? The editor is disturbed by Dr. Wilson's narrowing the scope of arithmetic. He believes that arithmetic should be so learned that important principles are established and that these should be used in a variety of situations. This process of generalization is a part of the program in arithmetic. He agrees that a problem about the ratio of corn to hogs may have less local significance in Maine than it does in Iowa but as a teacher he never felt it obligatory to use all the problems in a book for all the pupils. Read Dr. Wilson's statement carefully and see if you agree and if you disagree be sure that you can defend your own views.

Understanding Meanings in Arithmetic

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DURING the last twenty years a great deal has been written about "meaningful arithmetic." Although there is some confusion over the terminology surrounding meaningful arithmetic as well as difference of opinion as to what constitutes a program of meaningful arithmetic, the term itself has been generally accepted by teachers and administrators. There is general agreement, expressed in the literature, that the basic meanings in arithmetic include the understanding of counting, place value, the decimal number system, the meaning of addition, subtraction, multiplication, and division with integers, fractions, and decimals, and the meaning of percentage.

It is also generally agreed, according to the literature, that arithmetic is a thought process involving generalization, organization, integration and reorganization of experience. This thinking about meaningful arithmetic has emphasized a need for a change in the concept of teaching. The role of the teacher has become one of motivating and stimulating the child to discover the meanings in arithmetic. Instead of requiring the child to follow set patterns without reasons or explanations the teacher provides a learning situation in which the child gains insight and understanding. Such modern procedures have led to a change in the concept of drill which should come after understanding has been developed and which should not be used prematurely as a method of teaching the processes in arithmetic. Drill that comes after the meanings have been established may well be called meaningful practice and should be used by the teacher.

An observation of the practices of teachers reveals that many teachers give only lip service to the idea of meaning in arithmetic. According to Van Engen:

> The word "meaning" has found its way into the language of the arithmetic, in the past decade, with ever increasing frequency. Like many words, which are used frequently, there seems to be evidence that all teachers really do not interpret the "meaning" of meanings in the same way when applied to arithmetic. As evidence, read the protests found in the literature, against those teachers who say they are teaching arithmetic meaningfully, but according to numerous authors, these same teachers are only good "drill sergeants." Briefly described it is often heard said that "teachers give lip service to meaning."[1]

Administrators often stress the importance of achieving prescribed norms as measured by standardized arithmetic tests. The result has been an emphasis, and often

[1] H. Van Engen, "A Summary of Research and Investigations and Their Implications for the Organization and Learning of Arithmetic," *Mathematics Teacher*, 41: 260, Oct. 1948.

an over-emphasis, upon the development of computational skill. Some justify this in the belief that computational skill leads to understanding.

This writer conducted a study to determine the relationship between computational skill and the understanding of the meanings in arithmetic.[2]

Design of the Study

The *California Arithmetic Test*, Intermediate Form, for Grades 7-8-9 was administered to 271 7B pupils and to 110 8B pupils. Only the computation parts of the test, namely, Test 4, Sections E, F, G and H were used in order to determine the degree of computational skill attained by the subjects of this study. A section was devoted to each of the four fundamental processes of addition, subtraction, multiplication, and division. Although the test directions allowed 10 minutes for each of the first two sections and 12 minutes for each of the last two sections, the pupils were allowed an extra 16 minutes at the conclusion of the prescribed time because the purpose of administering this test was to determine the computational skill, and not the grade placement, achieved by the pupils. Whereas only a few pupils were able to complete the test in the original time allotted, only two or three were unable to complete the test with the additional time allowed.

After all of the test papers were scored, the pupils were classified into three groups. Group I was composed of those pupils whose scores were above the mean of the total sample plus one standard deviation. Group II was composed of the pupils whose scores fell between the mean minus one standard deviation and the mean plus one standard deviation. Group III was composed of those whose scores were below the mean minus one standard deviation. The seventh and eighth grade pupils were treated as two separate samples.

About three to four weeks after the computation test was given, the children were administered the writer's original meanings test, *A Test On Basic Concepts and Processes of Arithmetic*.[3] This test of 16 basic concepts was composed of three parts. Part I contained 48 exercises to test the basic ideas of arithmetic. Some of the exercises have only one part while others have a varying number of parts. An exercise was marked correct only if all of the parts were answered correctly. Any incorrect response in a series of steps was considered to be an indication that the child did not have complete understanding of the process performed. Part II was composed of 17 exercises that repeated the basic ideas of Part I, but on a more difficult or advanced level. Part III contained only 7 exercises that tested the thought processes of Parts I and II, but these exercises were considered to be more difficult than the corresponding exercises in the other parts. For the purpose of administering the test Parts II and III were combined as one part and were called Part II of the meanings test. The pupils were allowed 45 minutes for Part I and 30 minutes for Part II with a ten minute recess after Part I was administered.

Coefficients of correlation between the standardized computation test and the writer's meanings test were calculated for the total sample and for each of the three groups classified according to the achievement on the computation test.

Findings

1. The seventh grade pupils had a mean score of 51.87 (representing 65% mastery) on the *California Arithmetic Test* and a standard deviation of 14.85. The pupils that rated high had scores above 66.72 (51.87+14.85); the average group had scores between 66.72 and 37.02 (51.87±14.85), and the low group had scores below 37.02. The high group successfully answered at least 83 per cent of the 80 items that composed the computation test. Of the 51 pupils in this group, 20 had scores of 72 or higher, a 90 per cent

[2] David Rappaport, "An Investigation of the Degree of Understanding of Meanings in Arithmetic of Pupils in Selected Elementary Schools," (unpublished Doctor's dissertation, Northwestern University, Evanston, Illinois, 1957).

[3] A specimen copy of this test is included in the author's dissertation which can be read in the library of Northwestern University. A limited number of copies will be mailed to interested persons upon request.

achievement, while 10 had scores of 75 or higher, a 95 per cent achievement. This shows that the top group had achieved high computational skill, assuming that 90 per cent is considered high achievement.

The eighth grade pupils had a mean score of 62.12 on the *California Arithmetic Test* and a standard deviation of 12.29. The pupils who rated high had scores above 74.41 (62.12+12.29), the average group had scores between 74.41 and 49.83 (62.12 −12.29), and the low group had scores below 49.83. The mean of 62.12 represented approximately 78 per cent of the 80 items. Pupils whose scores were 74.41 or above had an achievement of 93 per cent or higher. Of the 15 pupils in the high group, 8 scored at least 78 and 2 pupils had perfect scores of 80. The high group of these eighth graders had achieved very high computational skill.

2. The seventh grade pupils achieved a mean of 26.98 on the writer's Meanings Test and a standard deviation of 11.90. This indicates that the pupils of this study averaged 37 per cent (27 out of 72 items) understanding of the basic concepts and processes in arithmetic. Only 8 of the 299 pupils (this number, 299, includes 28 who did not take the computation test and were, thus, not included in the correlations) showed an understanding of 75 per cent or higher and only 63 showed an understanding of 50 per cent or better. The lack of understanding of the meanings in arithmetic achieved by the pupils of this study becomes more apparent by the fact that 236, or 79 per cent of the children, had an understanding below 50 per cent.

The eighth graders achieved a mean of 30.42 on the Meanings Test and a standard deviation of 12.50. Almost 74 per cent of this group achieved less than 50 per cent on this test.

3. The coefficients of correlation between the computation and meanings tests were as follows:

a. Seventh Graders
   (1) .63 for the group as a whole
   (2) .54 for the high group
   (3) .39 for the average group
   (4) .31 for the low group.

b. Eighth Graders
   (1) .63 for the group as a whole
   (2) .19 for the high group
   (3) .40 for the average group
   (4) .18 for the low group.

Conclusions

The following conclusions were drawn from this study:

1. The seventh and eighth grade pupils did not have an adequate understanding of the meanings in arithmetic, assuming that a score below 50 per cent was an indication of inadequacy.

2. Computational skill was not an indication of the understanding of the meanings of the processes used in the computation. Some pupils with very high computation scores achieved very low scores on the meanings test.

3. Pupils tended to follow mechanical processes rather than meaningful interpretations. They answered questions that were not required by doing work in the margins of the test papers although they were specifically instructed not to do any calculations. In the

TABLE

	Mean	Standard Deviation	Group I	Group II	Group III
California Arithmetic Test, Seventh Grade	51.87	14.85	66.72+	37.02–66.72	37.02–
California Arithmetic Test, Eighth Grade	62.12	12.29	74.41+	49.83–74.41	49.83–
Meanings Test Seventh Grade	26.98	11.90			
Meanings Test Eighth Grade	30.42	12.50			

test itself some exercises were worked out completely. The pupil was merely asked to explain the steps that had been performed. Yet the pupils often worked out the exercises in the margin and arrived at the answer already given instead of explaining the steps needed to derive the answer.

4. Pupils showed a greater concern over the derivation of answers to arithmetical exercises than in understanding the meanings in arithmetic.

Recommendations

The results of this study indicate a wide discrepancy between computational skill and understanding. Teachers and school administrators should make every effort to narrow this gap between skill and understanding. The following are just a few suggestions that would help children develop understanding of the basic meanings in arithmetic.

1. Teachers are sometimes unaware of what meanings in arithmetic should be learned by the pupils. Administrators, as well as teachers, should have an understanding of the basic meanings in arithmetic, and they should see that these meanings are stated clearly and interpreted in the course of study, outlines, or curriculum guides. These meanings are often implied in the curriculum guides but this implication is not sufficient to make the teacher aware of the specific meanings.

2. Teachers should examine their own understanding of the basic meanings in arithmetic. Many teachers have learned their arithmetic as a skill subject by methods of drill. Such teachers should attend some classes at teacher training institutions or else, the administrator should establish an in-service training program.

3. Teachers should motivate, stimulate, and create a learning atmosphere in which pupils discover for themselves the meanings in arithmetic. They will have to move more slowly, especially the teachers in the primary grades, in order to give the pupils time and opportunities in which to make these discoveries. Many teachers rush to the abstract levels of arithmetic too soon. They should show a greater concern for the thought processes involved in the solution of arithmetic exercises. Children must learn to do accurate work and will, therefore, need a great deal of practice. But practice should come after understandings have been developed.

4. Some school administrators and teachers have been satisfied with the mere development of computational skills. Arithmetic then becomes a tool subject and drill becomes the method of instruction. This study has shown that skills are not enough, that understandings are needed. Some teachers have refrained from emphasis on meanings because they fear that their pupils will make slow progress. They fear that their class will not do as well on standardized arithmetic tests. The misuse of standardized tests has been treated in the literature and will not be discussed here.

The writer hopes that the conditions described in this study will help teachers and administrators to become more aware of the discrepancy between computational skill and understanding of the basic meanings in arithmetic and that they will take steps, where such are necessary, to develop a program of meaningful arithmetic.

EDITOR'S NOTE. Children tend to learn just about that which the schools honestly try to teach. Dr. Rappaport's sample contained pupils who had learned computations rather well but were not so well grounded in the understandings which he measured. The comparatively low correlations between these two factors shows that teaching for the one does not insure the other. The pupils tended to mechanical processes. One asks whether the variables might not have been better chosen. For example, let one be degree of understanding of a process such as division and the other the degree of competence in computations of division. This would be a more acceptable argument than one derived from a general test of understanding and one of computations.

Dr. Rappaport feels that administrators, supervisors and teachers should have a clear understanding of the basic meanings in arithmetic, and that these should be clearly stated in courses of study, curriculum guides, and outlines; that teachers should motivate, stimulate, and create a learning atmosphere in which pupils discover for themselves the meanings in arithmetic; that children need much practice to do accurate work but this should come AFTER meanings have been developed and not before—in other words, MEANINGFUL PRACTICE instead of meaningless drill.

A Short Method of Long Division

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I. When the divisor is composed of two digits:

I will write down a small number first, using the divisor 12.

The second example will have a divisor with three digits, and the third will have a divisor with four digits.

EXAMPLE #1

LEFT		RIGHT
12	2475.03	206.25
	75	
	30	
	63	
	3	

Twelve into 24 goes two times. Put the figure 2 in the right hand block. Say two by 12 is 24, and nothing is 24. Bring down 7 under the 7. Say 12 into 7 goes no times. Put a zero next to the two in the right hand block. Bring down 5 under 5. You now have the figure 75. 12 goes into 75 six times plus. Now put 6 in the right hand block. Now say 6 times 12 is 72 and 3 is 75, put 3 under the above, and bring down zero. You now have the figure 30. 12 goes into 30 two times plus. Put the figure 2 next to six in the right hand block. Now you say 2 times 2 is 4, and 4 from 10 leaves 6. Put six under the zero above, then say 2 times 1 is 2, and one to carry* is 3, and 3 from 3 leaves nothing. Now bring the figure 3 above down and you have the figure 63. Twelve goes into 63 five times plus. Put 5 in the right hand block. Now say 5 times 2 is ten. Ten from 13 leaves 3. Put the 3 under the above 3. Then say 5 times one is 5, and one to carry is 6, and 6 from six leaves nothing. You now have the figure 3 which is the remainder or 3/12 or 1/4. The total answer is 206.25 and 1/4 cent.

* "Carry figures" may be inserted if needed.

II. When the divisor is composed of three digits:

EXAMPLE #2

LEFT		RIGHT
250	481,509	1926 9/250
	231 5	
	6 50	
	1 509	
	0 009	

250 into 481 goes one times plus. Now say one times zero is zero. Zero from 1 leaves one. Put down one. One times 5 is 5. 5 from 8 is 3. Put down 3 under 8. One times two is two. Put 2 under 4. You now have the figure 231. 250 won't go into 231, so bring down the above 5. You now have the figure 2315. 250 goes into 2315 nine times plus. Nine times zero is zero, and zero from 5 leaves 5. Put 5 under 5. Nine times 5 is 45, and 6 is 51. Put 6 under the one. Nine times 2 is 18, and 5 to carry is 23, so there is nothing to put down under 23. You now have the number 65 which can't be divided by 250, so bring down the above zero. You now have the figure 650. Divide 650 by 250. 250 goes into 650 2 times plus. 2 times zero is zero, and zero from zero is zero. Put zero under zero. Two times 5 is 10 and 10 from 15 leaves 5. Put down 5. Two times two is four and one to carry is 5. Five from 6 leaves one. Put down one. You now have 150. 250 won't go into 150, so bring down the above figure 9. Six times 5 is 30, and nothing is 30. Bring down zero. Six times 2 is 12, and 3 to carry is 15. Fifteen from 15 leaves nothing. Bring down 2 zeros. You have the remainder 0009. The correct answer is 1926 and 9/250.

III. When the divisor is composed of four digits:

EXAMPLE #3

LEFT		RIGHT
3695	579821	156 and 3401 remainder
	21032	
	25571	
	3401	

3695 goes into 5798 one times plus. One times 5 is 5, and 3 is 8. Put 3 under 8. One times 9 is 9 and nothing is 9. Put zero under 9. One times six is six and 1 is 7. Put 1 under 7. One times 3 is 3 and 2 is 5. You now have the number 2103. Now bring down the above 2. The number is 21032. 3695 goes into 21032 five times plus. Five times 5 is 25 and 7 is 32. Put down 7. Five times 9 is 45 and 3 to carry is 48 and 5 is 53. Put down 5. 5 times 6 is 30 and 5 to carry is 35. 35 from 40 leaves 5. Put down 5. Five times 3 is 15 and 4 is 19. Nineteen from 21 leaves 2. Put down 2. You now have 2557. Bring down the above one and you have the number 25571. Now divide 25571 by 3695. 3695 goes into 25571 6 times plus. Six times 5 is 30, and one is 31. Put down 1. Six times 9 is 54 and 3 to carry is 57, and nothing makes 57. Put down zero. Six times 6 is 36 and 5 is 41. 41 from 45 leaves 4. Put down 4. Six times 3 is 18 and 4 to carry is 22. Twenty-two from 25 leaves 3. Put down 3. You now have the number 3401 which is the remainder. Correct answer is 156 and 3401/3695.

The following example is proof that the answer is correct:

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  3695
 ×156
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 22170
 18475
  3695
------
576420
  3401  Remainder
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579,821  Correct Answer.
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EDITOR'S NOTE. Readers may wish to have certain of their more able pupils try this method of division which omits some of the pencil work of the conventional method. They will recall a similarity with synthetic division which they may have learned in algebra. The historical development of division involves many interesting frames and devices. The reader will find many of these illustrated in Smith's History of Mathematics, Volume II, published by Ginn and Company.

Illinois Council Meetings

In order that more people can attend, the Illinois Council of Teachers of Mathematics is holding six sectional meetings this spring. Each meeting has a section devoted to arithmetic and under the guidance of a capable leader. Among the well known speakers are Dr. Nathan Lazar, Dr. Kenneth May, Dr. Robert Pingry, Dr. Donavon Johnson, Dr. Philip Jones, and Dr. Alice Hach. Detailed programs for the various sections may be had from Dr. Francis R. Brown of Illinois State Normal University at Normal, Illinois. The dates for the meetings are given below.

SATURDAY, MARCH 15
Bellerville Township High School, Bellerville

SATURDAY, MARCH 22
Hinsdale Township High School, Hinsdale

SATURDAY, MARCH 22
Western Illinois University, Macomb

WEDNESDAY, APRIL 16
Eastern Illinois University, Charleston

SATURDAY, APRIL 19
Southern Illinois University, Carbondale

SATURDAY, APRIL 19
Illinois State Normal University, Normal

BOOK REVIEW

Counting and Arithmetic in the Infants School. Bass, W. G. and Dowty, O. S. London, England: George G. Harrap and Company, Ltd.

This book describes the method of instruction and ideas developed in the English Infant School where ages range from five to seven years.

It is interesting to note variations in procedures between those practices in England and in the United States. Thus, the book is primarily informational to the reader. As for evaluating the merits of either program, some kind of comparative study would be necessary.

The reviewer then will attempt to identify some of the differences as she observes them. Where in many instances attention is first given to the collection idea, the cardinal idea of number, and which seems to us an effective development of the concept of number in American schools, procedures in this book place emphasis on the series idea. All counting and processes are developed using the number line, rather than collections of groups of objects. Furthermore, to develop any new idea, the following sequence of activities is followed. There is first an informal teacher-demonstration. This is followed by demonstrations on the part of the children, and finally, children have materials where each may work with the idea using his own materials. The ideas are structured for the children. However, children are encouraged, according to this reference, to discover patterns. We contrast this with our practice of discovery and generalization.

The action of processes receives particular emphasis just as they are given in America. However, where we think of processes as regroupings in combining or separating collections, in the English Schools addition is associated with going to the number line, subtraction as coming from the number line, multiplication as going to the number line in groups, and division as coming from the number line in groups.

Too, instead of so-called uneven division being expressed as so many groups and a remainder, as $2\overline{)9}$ being 4 groups of 2 and a remainder of 1, or just 4 and a remainder of 1, they would explain this as 4 groups of 2 and 1 part of a group of 2. Although they write the answer $4\frac{1}{2}$, the '$\frac{1}{2}$' is not referred to as a fraction. Furthermore, they use the notation, $\frac{10}{(2)}$, where the '2' below the line and in parentheses means how many groups of 2 in 10.

Other variations in content are the attention given to directed counting, to unit fractions, to money, and the interchange of letters and numbers. Counting steps forwards and backwards directs attention to directed counting which in turn enables them to do some elementary work with negative numbers with young children. Again, the number line is the device used to develop these ideas. The curve of unit fractions and the "Convecta Fraction Board" are used effectively in developing the concepts of fractions—relationships, equivalence, and operations. The Counterplay Board for each child is a useful device following the class demonstrations by teacher and children in developing the ideas of the four fundamental operations with the counting numbers. Emphasis is placed on relationship between monetary coins by adding, subtracting, multiplying and dividing and changing from one to the other where necessary, rather than by such experiences of making change, etc. Interchange of letters and number may have much merit for many of our primary children in the States. Children in England learn to substitute letters for numbers as in c 3 t and numbers for letters 1, 2, 3, n, 5.

This book gives us insight into the method of instruction used in English schools in developing arithmetical ideas. We find differences when comparing these methods and those being used in American schools. Which approach is better? We would need to make some measures of learning results. The reviewer believes that there is merit in aspects of both programs.

E. GLENADINE GIBB

Something New to Try

Many youngsters who have asked me if there was any way of making long division palatable have discovered in this method an end to their former frustration and often surprisingly enough after a short trial period find they can endure and even like long division.

I recommend that teachers and students who try this technique do at least a dozen problems alternately with the traditional method before they discard the new operation because it is new and therefore suspect

Sample 1

Key		144
1—	12)	1728
2—	24	12
4—	48	52
8—	96	48
		—
Check—	120	48
		48
		—

Sample 2

Key	Dbls.	256
1 —	179)	45824
2 —	358	358
4 —	716	1002
8 —	1432	895
Check —	1790	1074
		1074

First the divisor is doubled three times, with (1) being placed next to the divisor, (2) placed next to the first doubling, (4) next to the 2nd doubling, and (8) next to the last doubling; meaning respectively 1, 2, 4, and 8 times the divisor.

To test for 3 times the divisor we add the doublings opposite the key numbers for 1 and 2; to test for 5 times the divisor we add the doublings opposite the key numbers 1 and 4; for trial (6) use keys 2 and 4; for trial (7) use the sum of keys 1, 2 and 4 or 1 from 8; for trial of 9 times the divisor use the doublings opposite 1 and 8.

We check the results of our doublings by adding the doublings opposite the key numbers 2 and 8 which should give us ten times the original divisor; if it does not, we must do our doubling again.

We now proceed easily with our long division in the usual way with the aid of so many known and accurate key trial divisors, eliminating guesswork. No multiplication tables are necessary beyond the two's table, quite a boon to the youngster who does not know his tables and to the teacher who has the double complication and frustration of trying to teach long division to students not knowing their tables. Most youngsters learned the two's table before other mathematical obstacles had caused them to give up in the arithmetic field as being a field too difficult for their intellects. This method of long division may well salvage the self respect of many of these students enabling them to go on in the field of mathematics.

Contributed by
ARNOLD F. TORRANCE
Garden City, N. Y.

Dress up Your Graphs

The teaching unit on graphing in junior high school grades is sometimes a source of great inspiration to me.

If I am to teach the vertical bar, the broken line, the circle, or the pictograph type, I always try to use some clever ideas in different ways to attract interest and attention to the title of the graph as well as the information it contains.

The following is a list of things I suggest that students try out and they do it with great success.

1. Always use color. Coloring pencils, water colors, or crayons for printing titles, numbers, drawing or sketching symbols.

2. Use geometric patterns to fill in vertical bars or horizontal bars or to draw symbols in pictographs (triangular, circular, rectangular).

3. Use common symbols to show certain ideas . . . clouds, sun, rain, snow, hail for

WEATHER GRAPHS (temperature studies).

If your title suggests school activities such as sports, games, hobbies, stamps, areas of countries, depths of oceans or lakes, desserts eaten, or foods in general, STRESS USE OF THEIR SYMBOLS.

Many students will claim they cannot draw these ideas accurately; some sketch; others stress authenticity by pasting cut-out pictures from magazines of radios, TV sets, or actually use a part of their stamp collection and paste them on the graph paper.

During all of this embellishment, I still stress the fundamental ideas of the mathematics involved. These ideas follow AFTER our discussion of the planning of the number scale, measuring after rounding off, etc.

4. Stress Originality. Use the topic of FAVORITES in developing titles for graphs. Favorite Sports, Desserts, Meats Eaten, Books to Read, School Subjects, Types of Music, TV Programs, Colors, or Clothing Worn. You soon discover you cannot wear out the list of favorite ideas some seventh and eighth graders have on many topics.

I hope I have given some teachers a few ideas so that the next time they teach the graphing unit their results will become works of art in mathematics and that they will have much good material for display purposes.

GEORGE JANICKI
Elmwood Park, Ill.

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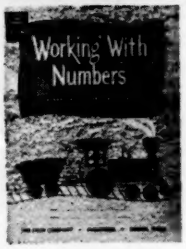

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